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SYMMETRIC HOMOLOGY OF ALGEBRAS: FOUNDATIONS

SHAUN V. AULT

Abstract. Symmetric homology of a unital algebra A over a commutative ground ring k is defined using derived functors and the symmetric bar construction of Fiedorowicz. If $A = k[G]$ is a group ring, then $HS_*(k[G])$ is related to stable homotopy theory. Two chain complexes that compute $HS_*(A)$ are constructed, both making use of a symmetric monoidal category ΔS_+ containing ΔS. Two spectral sequences are found that aid in computing symmetric homology. In the second spectral sequence, a complex isomorphic to the suspension of the cycle-free chessboard complex Ω_{p+1} of Vrećica and Živaljević plays an important role. Recent results on the connectivity of Ω_n imply finite-dimensionality of the symmetric homology groups of finite-dimensional algebras.

1. Introduction and Definitions

The theory of symmetric homology, in which the symmetric groups Σ_k^{op}, for $k \geq 0$, play the role that the cyclic groups do in cyclic homology, begins with the definition of the category ΔS, containing the simplicial category Δ as subcategory. Indeed, ΔS is an example of *crossed simplicial group* [8].

1.1. **The category ΔS.** Let ΔS be the category that has as objects, the ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and as morphisms, pairs (ϕ, g), where $\phi : [n] \to [m]$ is a non-decreasing map of sets (*i.e.*, a morphism in Δ), and $g \in \Sigma_{n+1}^{\mathrm{op}}$ (the opposite group of the symmetric group acting on $[n]$). The element g represents an automorphism of $[n]$, and as a set map, takes $i \in [n]$ to $g^{-1}(i)$. Equivalently, a morphism in ΔS is a morphism in Δ together with a total ordering of the domain $[n]$. Composition of morphisms is achieved as in [8], namely, $(\phi, g) \circ (\psi, h) := (\phi \cdot g^*(\psi), \psi^*(g) \cdot h)$.

Remark 1. Observe that the properties of $g^*(\phi)$ and $\phi^*(g)$ stated in Prop. 1.6 of [8] are formally rather similar to the properties of exponents. Indeed, the notation $g^\phi := \phi^*(g), \phi^g := g^*(\phi)$ will generally be used in lieu of the original notation in what follows.

For each n, let τ_n be the $(n+1)$-cycle $(0, n, n-1, \ldots, 1) \in \Sigma_{n+1}^{\mathrm{op}}$. The subgroup generated by τ_n is isomorphic to the cyclic group $C_{n+1}^{\mathrm{op}} = C_{n+1}$. Indeed, we may view the cyclic category ΔC as the subcategory of ΔS consisting of all objects $[n]$ for $n \geq 0$, together with those morphisms of ΔS of the form (ϕ, τ_n^i) (cf. [12]). In this way, we get a natural chain of inclusions, $\Delta \hookrightarrow \Delta C \hookrightarrow \Delta S$.

It is often helpful to represent morphisms of ΔS as diagrams of points and lines, indicating images of set maps. Using these diagrams, we may see clearly how ψ^g and g^ψ are related to (ψ, g) (see Figure 1).

An equivalent characterization of ΔS comes from Pirashvili (cf. [19]), as the category $\mathcal{F}(\mathrm{as})$ of 'non-commutative' sets. The objects are sets $\underline{n} := \{1, 2, \ldots, n\}$ for $n \geq 0$. By convention, $\underline{0}$ is the empty set. A morphism in $\mathrm{Mor}_{\mathcal{F}(\mathrm{as})}(\underline{n}, \underline{m})$ consists of a map (of sets) $f : \underline{n} \to \underline{m}$ together with a total ordering Π_j on $f^{-1}(j)$ for all $j \in \underline{m}$. In such a case, denote by Π the partial order generated by all Π_j. If $(f, \Pi) : \underline{n} \to \underline{m}$ and $(g, \Psi) : \underline{m} \to \underline{p}$, their composition will be (gf, Φ), where Φ_j is the total ordering on $(gf)^{-1}(j)$ (for all $j \in \underline{p}$) induced by Π and Ψ. Explicitly, for each pair $i_1, i_2 \in (gf)^{-1}(j)$, we have $i_1 < i_2$ under Φ if and only if $[f(i_1) < f(i_2)$ under $\Psi]$ or $[f(i_1) = f(i_2)$ and $i_1 < i_2$ under $\Pi]$.

There is an obvious inclusion of categories, $\Delta S \hookrightarrow \mathcal{F}(\mathrm{as})$, taking $[n]$ to $\underline{n+1}$, but there is no object of ΔS that maps to $\underline{0}$. It will be useful to define $\Delta S_+ \supset \Delta S$ which is isomorphic to $\mathcal{F}(\mathrm{as})$:

Definition 2. ΔS_+ is the category consisting of all objects and morphisms of ΔS, with the additional object $[-1]$, representing the empty set, and a unique morphism $\iota_n : [-1] \to [n]$ for each $n \geq -1$.

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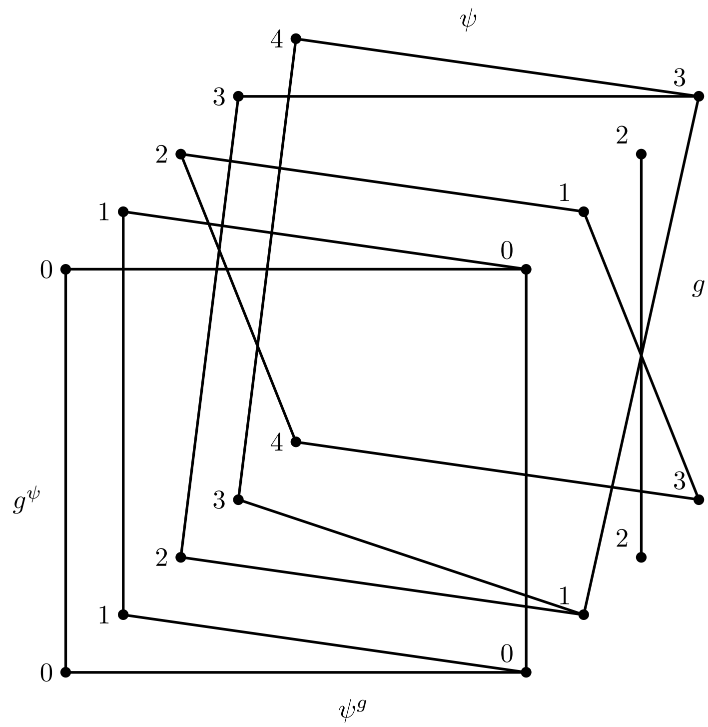

FIGURE 1. Morphisms of ΔS: $g \cdot \psi = \big(g^*(\psi), \psi^*(g)\big) = \big(\psi^g, g^\psi\big)$

Remark 3. Pirashvili's construction is a special case of a more general construction due to May and Thomason [17]. This construction associates to any topological operad $\{\mathcal{C}(n)\}_{n\geq 0}$ a topological category $\widehat{\mathcal{C}}$ together with a functor $\widehat{\mathcal{C}} \to \mathcal{F}$, where $\mathcal{F}$ is the category of finite sets, such that the inverse image of any function $f : \underline{m} \to \underline{n}$ is the space $\prod_{i=1}^{n} \mathcal{C}(\# f^{-1}(i))$. Composition in $\widehat{\mathcal{C}}$ is defined using the composition of the operad. May and Thomason refer to $\widehat{\mathcal{C}}$ as the *category of operators* associated to $\mathcal{C}$. They were interested in the case of an E_∞ operad, but their construction evidently works for any operad. The category of operators associated to the discrete A_∞ operad $\mathcal{A}ss$, which parametrizes monoid structures, is precisely Pirashvili's construction of $\mathcal{F}(as)$, *i.e.* ΔS_+.

One very useful advantage in enlarging our category to ΔS to ΔS_+ is the added structure inherent in ΔS_+.

Proposition 4. *ΔS_+ is a permutative category.*

Proof. Define the monoid product on objects by $[n] \odot [m] := [n+m+1]$, (disjoint union of sets), and on morphisms $(\phi, g) : [n] \to [n']$, $(\psi, h) : [m] \to [m']$, by $(\phi, g) \odot (\psi, h) = (\eta, k) : [n+m+1] \to [n'+m'+1]$, where (η, k) is just the morphism (ϕ, g) acting on the first $n+1$ points of $[n+m+1]$, and (ψ, h) acting on the remaining points.

The unit object will be $[-1] = \emptyset$. $\odot$ is clearly associative, and $[-1]$ acts as two-sided identity. Finally, define the transposition transformation $\gamma_{n,m} : [n] \odot [m] \to [m] \odot [n]$ to be the identity on objects, and on morphisms to be precomposition with the block transposition that switches the first block of size $n+1$ with the second block of size $m+1$. $\square$

Remark 5. The fact that ΔS_+ is permutative shall be exploited to prove that $HS_*(A)$ admits homology operations in a forthcoming paper.

For the purposes of computation, a morphism $\alpha : [n] \to [m]$ of ΔS may be conveniently represented as a tensor product of monomials in the formal non-commuting variables $\{x_0, x_1, \ldots, x_n\}$. Let $\alpha = (\phi, g)$, with $\phi \in \mathrm{Mor}_\Delta([n], [m])$ and $g \in \Sigma_{n+1}^{\mathrm{op}}$. The tensor representation of α will have $m+1$ tensor factors. Each

x_i will occur exactly once, in the order $x_{g(0)}, x_{g(1)}, \ldots, x_{g(n)}$. The i^{th} tensor factor consists of the product of $\#\phi^{-1}(i-1)$ variables, with the convention that the empty product will be denoted 1. Thus, the i^{th} tensor factor records the total ordering of $\phi^{-1}(i)$. As an example, the tensor representation of the morphism depicted in Fig. 2 is $x_0x_1 \otimes x_3x_4 \otimes 1 \otimes x_2$.

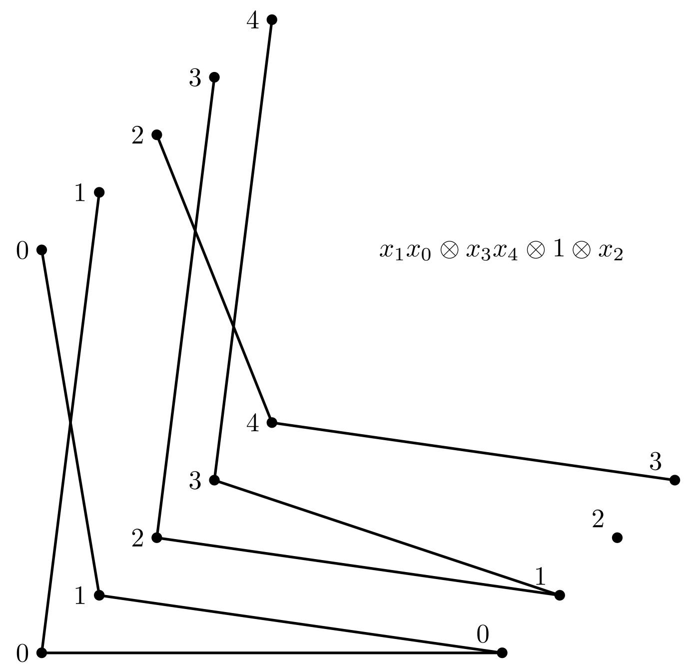

FIGURE 2. Morphisms of ΔS in tensor notation

With this notation, the composition of two morphisms $\alpha = X_0 \otimes X_1 \otimes \ldots \otimes X_m : [n] \to [m]$ and $\beta = Y_1 \otimes Y_2 \otimes \ldots Y_n : [p] \to [n]$ is given by, $\alpha\beta = Z_0 \otimes Z_1 \otimes \ldots \otimes Z_m$, where Z_i is determined by replacing each variable in the monomial $X_i = x_{j_1} \ldots x_{j_s}$ in α by the corresponding monomials Y_{j_k} in β. So, $Z_i = Y_{j_1} \ldots Y_{j_s}$.

1.2. **Homological Algebra of Functors.** Recall, for a category $\mathscr{C}$, a $\mathscr{C}$-module is covariant functor $F : \mathscr{C} \to k$-**Mod**. Similarly, a $\mathscr{C}^{\mathrm{op}}$-module is a contravariant functor $G : \mathscr{C} \to k$-**Mod**. Let M be a $\mathscr{C}^{\mathrm{op}}$-module and N be a $\mathscr{C}$-module. Following MacLane [13], define the tensor product of functors as a coend, $M \otimes_{\mathscr{C}} N := \int^X (MX) \otimes (NX)$. That is,

$$M \otimes_{\mathscr{C}} N := \bigoplus_{X \in \mathrm{Obj}\mathscr{C}} M(X) \otimes_k N(X)/\approx,$$

where the equivalence $\approx$ is generated by $y \otimes f_*(x) \approx f^*(y) \otimes x$ for $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$, $x \in N(X)$ and $y \in M(Y)$. The *trivial* $\mathscr{C}$-module, resp. $\mathscr{C}^{\mathrm{op}}$-module, denoted by $\underline{k}$ (for either variance), is the functor taking each object to k and each morphism to the identity.

Define a product on generators of $k[\mathrm{Mor}\mathscr{C}]$ by setting $f \cdot g := f \circ g$, if the maps are composable, and $f \cdot g := 0$ otherwise. Extending the product k-linearly, we obtain a ring structure on $R_{\mathscr{C}} := k[\mathrm{Mor}\mathscr{C}]$. In general $R_{\mathscr{C}}$ does not have a unit, only *local units*; that is, for any finite set of elements of $R_{\mathscr{C}}$, there is an element that acts as identity element for these elements. Observe, a $\mathscr{C}$-module M is equivalent to a left $R_{\mathscr{C}}$-module structure on $\bigoplus_{X \in \mathrm{Obj}\mathscr{C}} M(X)$.

In either characterization, it makes sense to talk about projective $\mathscr{C}$-modules and the torsion products $\mathrm{Tor}_i^{\mathscr{C}}(M,N)$ for $\mathscr{C}$-module M and $\mathscr{C}^{\mathrm{op}}$-module N. (It is also possible to define $\mathrm{Tor}_i^{\mathscr{C}}(M,N)$ directly as the derived functors of the categorical tensor product construction; see [8].)

Remark 6. Note, the existing literature based on the work of Connes, Loday and Quillen consistently defines the categorical tensor product in the reverse sense: $N \otimes_{\mathscr{C}} M$ is the direct sum of copies of $NX \otimes_k MX$ modded out by the equivalence $x \otimes f^*(y) \approx f_*(x) \otimes y$ for all $\mathscr{C}$-morphisms $f : X \to Y$. In this context, N is covariant, while M is contravariant. I chose to follow the convention of Pirashvili and Richter [20] in writing tensor products as $M \otimes_{\mathscr{C}} N$ so that the equivalence $\xi : \mathscr{C}\text{-}\mathbf{Mod} \to k[\mathrm{Mor}\mathscr{C}]\text{-}\mathbf{Mod}$ passes to tensor products in a straightforward way: $\xi(M \otimes_{\mathscr{C}} N) = \xi(M) \otimes_{k[\mathrm{Mor}\mathscr{C}]} \xi(N)$.

1.3. **The Symmetric Bar Construction.** Now that we have defined the category ΔS, the next step should be to define an appropriate bar construction. Recall that the *cyclic bar construction* is a functor $B_*^{cyc}A : \Delta C^{\mathrm{op}} \to k\text{-}\mathbf{Mod}$. We then take the groups $\mathrm{Tor}_i^{\Delta C}(B_*^{cyc}A, \underline{k})$ as our definition of $HC_i(A)$ for $i \geq 0$. However the results of [8] show that the cyclic bar construction does not extend to a functor $\Delta S^{\mathrm{op}} \to k\text{-}\mathbf{Mod}$. Furthermore, for *any* functor $F : \Delta S^{\mathrm{op}} \to k\text{-}\mathbf{Mod}$, the groups $\mathrm{Tor}_i^{\Delta S}(F, \underline{k})$ simply compute the homology of the underlying simplicial module of F (given by restricting F to Δ^{op}). The second author discovered [7] that there is a natural extension of the cyclic bar construction not to a *contravariant* functor on ΔS, but to a *covariant* functor.

Definition 7. Let A be an associative, unital algebra over a commutative ground ring k. Define a functor $B_*^{sym}A : \Delta S \to k\text{-}\mathbf{Mod}$ by:

$$B_n^{sym}A := B_*^{sym}A[n] := A^{\otimes(n+1)}$$

$$B_*^{sym}A(\alpha) : (a_0 \otimes a_1 \otimes \ldots \otimes a_n) \mapsto \alpha(a_0, \ldots, a_n),$$

where $\alpha : [n] \to [m]$ is represented in tensor notation, and evaluation at $(a_0, \ldots, a_n)$ simply amounts to substituting each a_i for x_i and multiplying the resulting monomials in A. If the pre-image $\alpha^{-1}(i)$ is empty, then the unit of A is inserted. Observe that $B_*^{sym}A$ is natural in A.

Remark 8. Fiedorowicz [7] defines the symmetric bar construction functor for morphisms $\alpha = (\phi, g)$, where $\phi \in \mathrm{Mor}\Delta([n],[m])$ and $g \in \Sigma_{n+1}^{\mathrm{op}}$, via

$$\begin{aligned} B_*^{sym}A(\phi)(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= \left(\prod_{a_i \in \phi^{-1}(0)} a_i\right) \otimes \ldots \otimes \left(\prod_{a_i \in \phi^{-1}(n)} a_i\right) \\ B_*^{sym}A(g)(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_{g^{-1}(0)} \otimes a_{g^{-1}(1)} \otimes \ldots \otimes a_{g^{-1}(n)} \end{aligned}$$

However, in order that this becomes consistent with earlier notation, we should require $B_*^{sym}A(g)$ to permute the tensor factors in the inverse sense:

$$B_*^{sym}A(g)(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_{g(0)} \otimes a_{g(1)} \otimes \ldots \otimes a_{g(n)}.$$

$B_*^{sym}A$ may be regarded as a simplicial k-module via the chain of functors, $\Delta^{\mathrm{op}} \hookrightarrow \Delta C^{\mathrm{op}} \xrightarrow{\cong} \Delta C \hookrightarrow \Delta S$. Here, the isomorphism $D : \Delta C^{\mathrm{op}} \to \Delta C$ is the standard duality (see [12]). Fiedorowicz showed in [7] that $B_*^{sym}A \circ D = B_*^{cyc}A$. Note that by duality of ΔC, it is equivalent to use the functor $B_*^{sym}A$, restricted to morphisms of ΔC in order to compute $HC_*(A)$.

Definition 9. The **symmetric homology** of an associative, unital k-algebra A is denoted $HS_*(A)$, and is defined as:

$$HS_*(A) := \mathrm{Tor}_*^{\Delta S}(\underline{k}, B_*^{sym}A)$$

Remark 10. Since $\underline{k} \otimes_{\Delta S} M \cong \mathrm{colim}_{\Delta S} M$, for any ΔS-module M, we can alternatively describe symmetric homology as derived functors of colim.

$$HS_i(A) = \operatorname*{colim}_{\Delta S}{}^{(i)} B_*^{sym}A.$$

(To see the relation with higher colimits, we need to tensor a projective resolution of $B_*^{sym}A$ with $\underline{k}$).

1.4. **The Standard Resolution.** Let $\mathscr{C}$ be a category. The rank 1 free $\mathscr{C}$-module is $k[\mathrm{Mor}\mathscr{C}]$, with the left action of composition of morphisms. Now as k-module, $k[\mathrm{Mor}\mathscr{C}]$ decomposes into the direct sum,

$$k[\mathrm{Mor}\mathscr{C}] = \bigoplus_{X \in \mathrm{Obj}\mathscr{C}} \left(\bigoplus_{Y \in \mathrm{Obj}\mathscr{C}} k\left[\mathrm{Mor}_{\mathscr{C}}(X,Y)\right] \right).$$

By abuse of notation, denote $\bigoplus_{Y \in \mathrm{Obj}\mathscr{C}} k\left[\mathrm{Mor}_{\mathscr{C}}(X,Y)\right]$ by $k\left[\mathrm{Mor}_{\mathscr{C}}(X,-)\right]$. So there is a direct sum decomposition as left $\mathscr{C}$-module),

$$k[\mathrm{Mor}\mathscr{C}] = \bigoplus_{X \in \mathrm{Obj}\mathscr{C}} k\left[\mathrm{Mor}_{\mathscr{C}}(X,-)\right].$$

Thus, the submodules $k\left[\mathrm{Mor}_{\mathscr{C}}(X,-)\right]$ are projective left $\mathscr{C}$-modules.

Similarly, $k[\mathrm{Mor}\mathscr{C}]$ is the rank 1 free right $\mathscr{C}$-module, with right action of pre-composition of morphisms, and as such, decomposes as:

$$k[\mathrm{Mor}\mathscr{C}] = \bigoplus_{Y \in \mathrm{Obj}\mathscr{C}} k\left[\mathrm{Mor}_{\mathscr{C}}(-,Y)\right]$$

Again, the notation $k\left[\mathrm{Mor}_{\mathscr{C}}(-,Y)\right]$ is shorthand for $\bigoplus_{X \in \mathrm{Obj}\mathscr{C}} k\left[\mathrm{Mor}_{\mathscr{C}}(X,Y)\right]$. The submodules $k\left[\mathrm{Mor}_{\mathscr{C}}(-,Y)\right]$ are projective as right $\mathscr{C}$-modules.

It will also be important to note that $k\left[\mathrm{Mor}_{\mathscr{C}}(-,Y)\right] \otimes_{\mathscr{C}} N \cong N(Y)$ as k-module via the evaluation map $f \otimes y \mapsto f_*(y)$. Similarly, $M \otimes_{\mathscr{C}} k\left[\mathrm{Mor}_{\mathscr{C}}(X,-)\right] \cong M(X)$.

Recall (*e.g.* in Quillen [21], Section 1), for an object $Y \in \mathrm{Obj}\mathscr{C}$, the category of *objects under* Y, denoted $Y \setminus \mathscr{C}$, has as objects all pairs (X, ϕ) such that $\phi \in \mathrm{Mor}_{\mathscr{C}}(Y,X)$, and as morphisms the commutative triangles. Define a contravariant functor $(- \setminus \mathscr{C})$ from $\mathscr{C}$ to **Cat** as follows: An object Y is sent to the under category $Y \setminus \mathscr{C}$. If $\nu : Y \to Y'$ is a morphism in $\mathscr{C}$, the functor $(\nu \setminus \mathscr{C}) : Y' \setminus \mathscr{C} \to Y \setminus \mathscr{C}$ is defined on objects by $(X, \phi) \mapsto (X, \phi\nu)$. Thus, $(- \setminus \mathscr{C})$ is a $\mathscr{C}^{\mathrm{op}}$-category.

As noted in [8], the nerve of $(- \setminus \mathscr{C})$ is a simplicial $\mathscr{C}^{\mathrm{op}}$-set, and the complex L_*, given by:

$$L_n := k\left[N(- \setminus \mathscr{C})_n\right]$$

is a resolution by projective $\mathscr{C}^{\mathrm{op}}$-modules of the trivial $\mathscr{C}^{\mathrm{op}}$-module, $\underline{k}$. Here, the boundary map is $\partial := \sum_i (-1)^i d_i$, where the d_i's come from the simplicial structure of the nerve of $(- \setminus \mathscr{C})$.

For completeness, we shall provide a proof of:

Proposition 11. *L_* is a resolution of $\underline{k}$ by projective $\mathscr{C}^{\mathrm{op}}$-modules.*

Proof. Fix $C \in \mathrm{Obj}\mathscr{C}$. Let $\epsilon : L_0(C) \to k$ be the map defined on generators by $\epsilon(C \to A_0) := 1_k$. The complex

$$k \xleftarrow{\epsilon} L_0(C) \xleftarrow{\partial} L_1(C) \xleftarrow{\partial} \ldots$$

is chain homotopic to the 0 complex via the homotopy,

$$h_{-1} : 1 \mapsto (C \xrightarrow{id} C)$$

$$h_n : (C \to A_0 \to \ldots \to A_n) \mapsto (C \xrightarrow{id} C \to A_0 \to \ldots A_n), \qquad \text{for } i \geq 0$$

Note that each $\mathscr{C}^{\mathrm{op}}$-module L_n is projective. □

Thus, we may compute $HS_*(A)$ as the homology groups of the following complex:

$$0 \longleftarrow L_0 \otimes_{\Delta S} B_*^{sym} A \longleftarrow L_1 \otimes_{\Delta S} B_*^{sym} A \longleftarrow L_2 \otimes_{\Delta S} B_*^{sym} A \longleftarrow \ldots \tag{1}$$

Denote the complex (1) by $\mathscr{Y}_* A$. Indeed, $\mathscr{Y}_* A$ is a simplicial k-module, and the assignment $A \mapsto \mathscr{Y}_* A$ is a functor k-**Alg** $\to$ k-**SimpMod**.

Corollary 12. *For an associative, unital k-algebra A, $HS_*(A) = H_*\left(\mathscr{Y}_* A;\, k\right)$. That is,*

$$HS_*(A) = H_*\left(k[N(- \setminus \Delta S)] \otimes_{\Delta S} B_*^{sym} A;\, k\right).$$

Remark 13. By remarks above, it is clear that the related complex $k[N(- \setminus \Delta C)] \otimes_{\Delta C} B_*^{sym} A$ computes $HC_*(A)$.

Remark 14. Observe that every element of $L_n \otimes_{\Delta S} B_*^{sym} A$ is equivalent to one in which the first morphism of the L_n factor is an identity:

$$[p] \xrightarrow{\alpha} [q_0] \xrightarrow{\beta_1} [q_1] \xrightarrow{\beta_2} \dots \xrightarrow{\beta_n} [q_n] \otimes (y_0 \otimes \dots \otimes y_p)$$

$$\approx [q_0] \xrightarrow{\mathrm{id}_{[q_0]}} [q_0] \xrightarrow{\beta_1} [q_1] \xrightarrow{\beta_2} \dots \xrightarrow{\beta_n} [q_n] \otimes \alpha_*(y_0, \dots, y_p)$$

Thus, we may consider $L_n \otimes_{\Delta S} B_*^{sym} A$ to be the k-module generated by the elements

$$\left\{ [q_0] \xrightarrow{\beta_1} [q_1] \xrightarrow{\beta_2} \dots \xrightarrow{\beta_n} [q_n] \otimes (y_0 \otimes \dots \otimes y_{q_0}) \right\},$$

where the tensor product is now over k. The face maps d_i are defined on generators by: d_0 omits β_1 and replaces $(y_0 \otimes \dots \otimes y_{q_0})$ by $\beta_1(y_0, \dots, y_{q_0})$; for $0 < i < n$, d_i composes β_{i+1} with β_i; and d_n omits β_n.

We now have enough tools to compute $HS_*(k)$. First, we need to show:

Lemma 15. *$N(\Delta S)$ is a contractible complex.*

Proof. Define a functor $\mathscr{F} : \Delta S \to \Delta S$ by

$$\mathscr{F} : [n] \mapsto [0] \odot [n],$$

$$\mathscr{F} : f \mapsto \mathrm{id}_{[0]} \odot f,$$

using the monoid multiplication $\odot$ defined above.

There is a natural transformation $\mathrm{id}_{\Delta S} \to \mathscr{F}$ given by the following commutative diagram for each $f : [m] \to [n]$:

$$\begin{tikzcd}
{[m]} \arrow[r, "f"] \arrow[d, "\delta_0"'] & {[n]} \arrow[d, "\delta_0"] \\
{[m+1]} \arrow[r, "\mathrm{id} \odot f"] & {[n+1]}
\end{tikzcd}$$

Here, $\delta_j^{(k)} : [k-1] \to [k]$ is the Δ morphism that misses the point $j \in [k]$.

Consider the constant functor $\Delta S \xrightarrow{[0]} \Delta S$ that sends all objects to $[0]$ and all morphisms to $\mathrm{id}_{[0]}$. There is a natural transformation $[0] \to \mathscr{F}$ given by the following commutative diagram for each $f : [m] \to [n]$.

$$\begin{tikzcd}
{[0]} \arrow[r, "\mathrm{id}"] \arrow[d, "0_0"'] & {[0]} \arrow[d, "0_0"] \\
{[m+1]} \arrow[r, "\mathrm{id} \odot f"] & {[n+1]}
\end{tikzcd}$$

Here, $0_j^{(k)} : [0] \to [k]$ is the morphism that sends the point 0 to $j \in [k]$.

Natural transformations induce homotopy equivalences (see [23] or Prop. 1.2 of [21]), so in particular, the identity map on $N(\Delta S)$ is homotopic to the map that sends $N(\Delta S)$ to the nerve of a trivial category. Thus, $N(\Delta S)$ is contractible. □

Corollary 16. *The symmetric homology of the ground ring k is isomorphic to k, concentrated in degree* 0.

Proof. $HS_*(k)$ is the homology of the chain complex generated (freely) over k by the chains

$$\left\{ [q_0] \xrightarrow{\beta_1} [q_1] \xrightarrow{\beta_2} \dots \xrightarrow{\beta_n} [q_n] \otimes (1 \otimes \dots \otimes 1) \right\},$$

where $\beta_i \in \mathrm{Mor}_{\Delta S}([q_{i-1}], [q_i])$. Each chain $[q_0] \to [q_1] \to \dots \to [q_n] \otimes (1 \otimes \dots \otimes 1)$ may be identified with the chain $[q_0] \to [q_1] \to \dots \to [q_n]$ of $N(\Delta S)$, and clearly this defines a chain isomorphism to $N(\Delta S)$. The result now follows from Lemma 15. □

2. Symmetric Homology with Coefficients

2.1. **Definitions and Conventions.** Following the conventions for Hochschild and cyclic homology in Loday [12], when we need to indicate explicitly the ground ring k over which we compute symmetric homology of A, we shall use the notation: $HS_*(A \mid k)$. Furthermore, since the notion "ΔS-module" does not explicitly state the ground ring, we shall use the bulkier "ΔS-module over k" when the context is ambiguous.

Definition 17. If $\mathscr{C}_*$ is a complex that computes symmetric homology of the algebra A over k, and M is a k-module, then the symmetric homology of A over k *with coefficients in* M is defined by $HS_*(A; M) := H_*(\mathscr{C}_* \otimes_k M)$.

Remark 18. Definition 17 is independent of the particular choice of complex $\mathscr{C}_*$, so we shall generally use the complex $\mathscr{Y}_*A = k[N(- \setminus \Delta S)] \otimes_{\Delta S} B_*^{sym}A$ of Cor. 12 in this section.

Proposition 19. *If M is flat over k, then $HS_*(A; M) \cong HS_*(A) \otimes_k M$.*

Proof. Since M is k-flat, the functor $- \otimes_k M$ is exact. □

Corollary 20. *For any $\mathbb{Z}$-agebra A, $HS_n(A; \mathbb{Q}) \cong HS_n(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

2.2. **Universal Coefficient Theorem for Symmetric Homology.** Our goal will be the proof of the following:

Theorem 21. *If A is a flat k-algebra, and B is a commutative k-algebra, then there is a spectral sequence with*

$$E_2^{p,q} := \mathrm{Tor}_p^k\left(HS_q(A \mid k), B\right) \Rightarrow HS_*(A; B).$$

To that end, we begin with some preliminary results.

Lemma 22.
i. If A is a flat k-algebra, then $\mathscr{Y}_nA$ is flat for each n.
ii. If A is a projective k-algebra, then $\mathscr{Y}_nA$ is projective for each n.

Proof. By Remark 14 we may identify:

$$\mathscr{Y}_nA \cong \bigoplus_{m \geq 0} \left(k[N([m] \setminus \Delta S)_{n-1}] \otimes_k A^{\otimes(m+1)} \right).$$

Note, $k[N([m] \setminus \Delta S)_{n-1}]$ is free, so if A is flat (resp. projective), then $\mathscr{Y}_nA$ is also flat (resp. projective). □

Proposition 23. *If B is a commutative k-algebra, then there is an isomorphism $HS_*(A \otimes_k B \mid B) \cong HS_*(A; B)$.*

Proof. Here, we are viewing $A \otimes_k B$ as a B-algebra via the inclusion $B \cong 1_A \otimes_k B \hookrightarrow A \otimes_k B$. Observe, there is an isomorphism

$$(A \otimes_k B) \otimes_B (A \otimes_k B) \xrightarrow{\cong} A \otimes_k A \otimes_k (B \otimes_B B) \xrightarrow{\cong} (A \otimes_k A) \otimes_k B.$$

Iterating this for n-fold tensors of $A \otimes_k B$,

$$\underbrace{(A \otimes_k B) \otimes_B \ldots \otimes_B (A \otimes_k B)}_{n} \cong \underbrace{A \otimes_k \ldots \otimes_k A}_{n} \otimes_k B.$$

This shows that the ΔS-module over B, $B_*^{sym}(A \otimes_k B)$ is isomorphic as k-module to $(B_*^{sym}A) \otimes_k B$ over k. The proposition now follows essentially by definition. Let L_* be the resolution of $\underline{k}$ by projective ΔS^{op}-modules (over k) given by $L_* = k[N(- \setminus \Delta S)]$. Taking tensor products (over k) with the algebra B, we obtain a projective resolution of the trivial ΔS^{op}-module over B. Thus,

$$HS_*(A \otimes_k B \mid B) = H_*\left((L_* \otimes_k B) \otimes_{B[\mathrm{Mor}\Delta S]} B_*^{sym}(A \otimes_k B);\ B\right) \tag{2}$$

On the chain level, there are isomorphisms:

$$\begin{aligned}(L_* \otimes_k B) \otimes_{B[\mathrm{Mor}\Delta S]} B_*^{sym}(A \otimes_k B) &\cong (L_* \otimes_k B) \otimes_{B[\mathrm{Mor}\Delta S]} (B_*^{sym}A \otimes_k B) \\ &\cong (L_* \otimes_{k[\mathrm{Mor}\Delta S]} B_*^{sym}A) \otimes_k B \qquad (3)\end{aligned}$$

The complex (3) computes $HS_*(A; B)$ by definition. □

Remark 24. Since $HS_*(A \mid k) = HS_*(A \otimes_k k \mid k)$, Prop. 23 allows us to identify $HS_*(A \mid k)$ with $HS_*(A; k)$.

The construction $HS_*(A; -)$ is a covariant functor, as is immediately seen on the chain level. In addition, $HS_*(-; M)$ is a covariant functor for any k-module, M.

Proposition 25. *Suppose $0 \to X \to Y \to Z \to 0$ is a short exact sequence of left k-modules, and suppose A is a flat k-algebra. Then there is an induced long exact sequence in symmetric homology:*

$$\ldots \to HS_n(A; X) \to HS_n(A; Y) \to HS_n(A; Z) \to HS_{n-1}(A; X) \to \ldots \tag{4}$$

Moreover, a map of short exact sequences, (α, β, γ), as in the diagram below, induces a map of the corresponding long exact sequences (commutative ladder)

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0 \end{tikzcd} \tag{5}$$

Proof. By Lemma 22, the hypothesis A is flat implies that the following is an exact sequence of chain complexes:

$$0 \to \mathscr{Y}_* A \otimes_k X \to \mathscr{Y}_* A \otimes_k Y \to \mathscr{Y}_* A \otimes_k Z \to 0.$$

This induces a long exact sequence in homology

$$\ldots \to H_n(\mathscr{Y}_* A \otimes_k X) \to H_n(\mathscr{Y}_* A \otimes_k Y) \to H_n(\mathscr{Y}_* A \otimes_k Z) \to H_{n-1}(\mathscr{Y}_* A \otimes_k X) \to \ldots$$

as required.

Now let (α, β, γ) be a morphism of short exact sequences, as in diagram (5). Consider the diagram,

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ HS_n(A; X) \arrow[r, "\alpha_*"] \arrow[d] & HS_n(A; X') \arrow[d] \\ HS_n(A; Y) \arrow[r, "\beta_*"] \arrow[d] & HS_n(A; Y') \arrow[d] \\ HS_n(A; Z) \arrow[r, "\gamma_*"] \arrow[d, "\partial"'] & HS_n(A; Z') \arrow[d, "\partial'"'] \\ HS_{n-1}(A; X) \arrow[r, "\alpha_*"] \arrow[d] & HS_{n-1}(A; X') \arrow[d] \\ \vdots & \vdots \end{tikzcd} \tag{6}$$

Since $HS_n(A; -)$ is functorial, the upper two squares of the diagram commute. Commutativity of the lower square follows from the naturality of the connecting homomorphism in the snake lemma. $\square$

Remark 26. Any family of additive covariant functors $\{T_n\}$ between two abelian categories is said to be a *long exact sequence of functors* if it takes short exact sequences to long exact sequences such as (4) and morphisms of short exact sequences to commutative ladders of long exact sequences such as (6). See [5], Definition 1.1 and also [18], section 12.1. The content of Prop. 25 is that for A flat, $\{HS_n(A; -)\}_{n \in \mathbb{Z}}$ is a long exact sequence of functors.

We may now prove Theorem 21.

Proof. Let $T_q : k\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ be the functor $HS_q(A; -)$. Observe, since A is flat, $\{T_q\}$ is a long exact sequence of additive covariant functors (Rmk. 26 and Prop. 25); $T_q = 0$ for sufficiently small q (indeed, for $q < 0$); and T_q commutes with arbitrary direct sums. Hence, by the Universal Coefficient Theorem of Dold (2.12 of [5]. See also McCleary [18], Thm. 12.11), there is a spectral sequence with $E_2^{p,q} := \mathrm{Tor}_p^k(T_q(k), B) \Rightarrow T_*(B)$. □

As an immediate consequence, we have the following result.

Corollary 27. *If $f : A \to A'$ is a k-algebra map between flat algebras which induces an isomorphism in symmetric homology, $HS_*(A) \xrightarrow{\cong} HS_*(A')$, then for a commutative k-algebra B, the map $f \otimes \mathrm{id}_B$ induces an isomorphism $HS_*(A; B) \xrightarrow{\cong} HS_*(A'; B)$.*

Under stronger hypotheses, the universal coefficient spectral sequence reduces to short exact sequences. Recall some notions of ring theory (c.f. the article Homological Algebra: Categories of Modules (200:K), Vol. 1, pp. 755-757 of [10]). A commutative ring k is said to have *global dimension* $\le n$ if for all k-modules X and Y, $\mathrm{Ext}_k^m(X, Y) = 0$ for $m > n$. k is said to have *weak global dimension* $\le n$ if for all k-modules X and Y, $\mathrm{Tor}_m^k(X, Y) = 0$ for $m > n$. Note, the weak global dimension of a ring is less than or equal to its global dimension, with equality holding for Noetherian rings but not in general. A ring is said to be *hereditary* if all submodules of projective modules are projective, and this is equivalent to the global dimension of the ring being no greater than 1.

Theorem 28. *If k has weak global dimension ≤ 1, then the spectral sequence of Thm. 21 reduces to short exact sequences,*

$$0 \longrightarrow HS_n(A \mid k) \otimes_k B \longrightarrow HS_n(A; B) \longrightarrow \mathrm{Tor}_1^k(HS_{n-1}(A \mid k), B) \longrightarrow 0. \tag{7}$$

Moreover, if k is hereditary and and A is projective over k, then these sequences split (unnaturally).

Proof. Assume first that k has weak global dimension ≤ 1. So $\mathrm{Tor}_p^k(T_q(k), B) = 0$ for all $p > 1$. Following Dold's argument (Corollary 2.13 of [5]), we obtain the required exact sequences,

$$0 \longrightarrow T_n(k) \otimes_k B \longrightarrow T_n(B) \longrightarrow \mathrm{Tor}_1^k(T_{n-1}(k), B) \longrightarrow 0.$$

Assume further that k is hereditary and A is projective. Then by Lemma 22, $\mathscr{Y}_n A$ is projective for each n. Theorem 8.22 of Rotman [22] then gives us the desired splitting. □

Remark 29. The proof given above also proves UCT for cyclic homology. A partial result along these lines exists in Loday ([12], 2.1.16). There, he shows $HC_*(A \mid k) \otimes_k K \cong HC_*(A \mid K)$ and $HH_*(A \mid k) \otimes_k K \cong HH_*(A \mid K)$ in the case that K is a localization of k, and A is a K-module, flat over k. I am not aware of a statement of UCT for cyclic or Hochschild homology in its full generality in the literature.

2.3. **Integral Symmetric Homology and a Bockstein Spectral Sequence.** We shall obtain a converse to Cor. 27 in the case $k = \mathbb{Z}$.

Theorem 30. *Let $f : A \to A'$ be an algebra map between torsion-free $\mathbb{Z}$-algebras. Suppose for $B = \mathbb{Q}$ and $B = \mathbb{Z}/p\mathbb{Z}$ for any prime p, the map $f \otimes \mathrm{id}_B$ induces an isomorphism $HS_*(A; B) \to HS_*(A'; B)$. Then f also induces an isomorphism $HS_*(A) \xrightarrow{\cong} HS_*(A')$.*

First, note that if A is flat over k, Prop. 25 allows one to construct the Bockstein homomorphisms $\beta_n : HS_n(A; Z) \to HS_{n-1}(A; X)$ associated to a short exact sequence of k-modules, $0 \to X \to Y \to Z \to 0$. These Bocksteins are natural in the following sense:

Lemma 31. *Suppose $f : A \to A'$ is a map of flat k-algebras, and $0 \to X \to Y \to Z \to 0$ is a short exact sequence of k-modules. Then the following diagram is commutative for each n:*

$$\begin{array}{ccc} HS_n(A; Z) & \xrightarrow{\beta} & HS_{n-1}(A; X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ HS_n(A'; Z) & \xrightarrow{\beta'} & HS_{n-1}(A'; X) \end{array}$$

Moreover if the induced map $f_* : HS_*(A; W) \to HS_*(A'; W)$ is an isomorphism for any two of $W = X$, $W = Y$, $W = Z$, then it is an isomorphism for the third.

Proof. A and A' flat imply both sequences of complexes are exact:

$$0 \to \mathscr{Y}_* A \otimes_k X \to \mathscr{Y}_* A \otimes_k Y \to \mathscr{Y}_* A \otimes_k Z \to 0,$$

$$0 \to \mathscr{Y}_* A' \otimes_k X \to \mathscr{Y}_* A' \otimes_k Y \to \mathscr{Y}_* A' \otimes_k Z \to 0.$$

The map $\mathscr{Y}_* A \to \mathscr{Y}_* A'$ induces a map of short exact sequences, hence induces a commutative ladder of long exact sequences of homology groups. In particular, the squares involving the boundary maps (Bocksteins) must commute.

Now, assuming further that f_* induces isomorphisms $HS_*(A; W) \to HS_*(A'; W)$ for any two of $W = X$, $W = Y$, $W = Z$, let V be the third module. The 5-lemma implies isomorphisms $HS_n(A; V) \xrightarrow{\cong} HS_n(A'; V)$ for each n. □

We shall now proceed with the proof of Thm. 30. All tensor products will be over $\mathbb{Z}$ in what follows.

Proof. Let A and A' be torsion-free $\mathbb{Z}$-modules. Over $\mathbb{Z}$, torsion-free implies flat. Let $f : A \to A'$ be an algebra map inducing isomorphism in symmetric homology with coefficients in $\mathbb{Q}$ and also in $\mathbb{Z}/p\mathbb{Z}$ for any prime p. For $m \geq 2$, there is a short exact sequence,

$$0 \longrightarrow \mathbb{Z}/p^{m-1}\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^m\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

Consider first the case $m = 2$. Since $HS_*(A; \mathbb{Z}/p\mathbb{Z}) \to HS_*(A'; \mathbb{Z}/p\mathbb{Z})$ is an isomorphism, Lemma 31 implies the induced map is an isomorphism for the middle term:

$$f_* : HS_*(A; \mathbb{Z}/p^2\mathbb{Z}) \xrightarrow{\cong} HS_*(A'; \mathbb{Z}/p^2\mathbb{Z}) \tag{8}$$

(Note, all maps induced by f on symmetric homology will be denoted by f_*.)

For the inductive step, fix $m > 2$ and suppose f induces an isomorphism in symmetric homology, $f_* : HS_*(A; \mathbb{Z}/p^{m-1}\mathbb{Z}) \xrightarrow{\cong} HS_*(A'; \mathbb{Z}/p^{m-1}\mathbb{Z})$. Again, Lemma 31 implies the induced map is an isomorphism on the middle term.

$$f_* : HS_*(A; \mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\cong} HS_*(A'; \mathbb{Z}/p^m\mathbb{Z}) \tag{9}$$

Denote $\mathbb{Z}/p^\infty\mathbb{Z} := \varinjlim \mathbb{Z}/p^m\mathbb{Z}$. Note, this is a *direct limit* in the sense that it is a colimit over a directed system. The direct limit functor is exact (Prop. 5.3 of [24]), so the maps $HS_n(A; \mathbb{Z}/p^\infty\mathbb{Z}) \to HS_n(A'; \mathbb{Z}/p^\infty\mathbb{Z})$ induced by f are isomorphisms, given by the chain of isomorphisms below:

$$HS_n(A; \mathbb{Z}/p^\infty\mathbb{Z}) \cong H_n(\varinjlim \mathscr{Y}_* A \otimes \mathbb{Z}/p^m\mathbb{Z}) \cong \varinjlim H_*(\mathscr{Y}_* A \otimes \mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{f_*}$$

$$\varinjlim H_*(\mathscr{Y}_* A' \otimes \mathbb{Z}/p^m\mathbb{Z}) \cong H_*(\varinjlim \mathscr{Y}_* A' \otimes \mathbb{Z}/p^m\mathbb{Z}) \cong HS_*(A'; \mathbb{Z}/p^\infty\mathbb{Z})$$

(Note, f_* here stands for $\varinjlim H_n(\mathscr{Y}_* f \otimes \mathrm{id})$.)

Finally, consider the short exact sequence of abelian groups,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \bigoplus_{p \text{ prime}} \mathbb{Z}/p^\infty\mathbb{Z} \longrightarrow 0$$

The isomorphism $f_* : HS_*(A; \mathbb{Z}/p^\infty\mathbb{Z}) \to HS_*(A'; \mathbb{Z}/p^\infty\mathbb{Z})$ passes to direct sums, giving isomorphisms for each n,

$$f_* : HS_n\left(A; \bigoplus_p \mathbb{Z}/p^\infty\mathbb{Z}\right) \xrightarrow{\cong} HS_n\left(A'; \bigoplus_p \mathbb{Z}/p^\infty\mathbb{Z}\right).$$

Together with the assumption that $HS_*(A; \mathbb{Q}) \to HS_*(A'; \mathbb{Q})$ is an isomorphism, another appeal to Lemma 31 gives the required isomorphism in symmetric homology, $f_* : HS_n(A) \xrightarrow{\cong} HS_n(A')$. □

Remark 32. Theorem 30 may be useful for determining integral symmetric homology, since rational computations are generally simpler, and computations mod p may be made easier due to the presence of additional structure.

Finally, we state a result along the lines of McCleary [18], Thm. 10.3. (This is a version of the Bockstein spectral sequence for symmetric homology.) Denote the torsion submodule of the graded module H_* by $\tau(H_*)$.

Theorem 33. *Suppose A is free of finite rank over $\mathbb{Z}$. Then there is a singly-graded spectral sequence with*

$$E^1_* := HS_*(A;\mathbb{Z}/p\mathbb{Z}) \Rightarrow HS_*(A)/\tau(HS_*(A)) \otimes \mathbb{Z}/p\mathbb{Z},$$

with differential map $d^1 = \beta$, the standard Bockstein map associated to $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$. Moreover, the convergence is strong.

The proof McCleary gives on p. 459 carries over to our case intact. All that is required for this proof is that each $H_n(\mathscr{Y}_*A)$ be a finitely-generated abelian group. The hypothesis that A is finitely-generated, coupled with a result of Section ?, namely Cor. 95, guarantees this. Note, over $\mathbb{Z}$, *free of finite rank* is equivalent to *flat and finitely-generated*.

3. Tensor Algebras

For a general k-algebra A, the standard resolution is often too difficult to work with. In order to develop workable methods to compute symmetric homology, it will become necessary to find smaller resolutions of $\underline{k}$. First, we develop the necessary results for the special case of tensor algebras. Indeed, tensor algebra arguments are also key in the proof of Fiedorowicz's Theorem (Thm. 1(i) of [7]) about the symmetric homology of group algebras.

Let $T : k\text{-}\mathbf{Alg} \to k\text{-}\mathbf{Alg}$ be the functor sending an algebra A to the tensor algebra generated by A, $TA := k \oplus A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \oplus \ldots$ There is an algebra homomorphism $\theta : TA \to A$, defined by multiplying tensor factors, $\theta(a_0 \otimes a_1 \otimes \ldots \otimes a_k) := a_0a_1 \cdots a_k$. In fact, θ defines a natural transformation $T \to \mathrm{id}$. We shall also make use of a k-module homomorphism h, sending the algebra A identically onto the summand A of TA. This map is a natural transformation from the forgetful functor $U : k\text{-}\mathbf{Alg} \to k\text{-}\mathbf{Mod}$ to the functor UT. The resolution of A by tensor algebras may be regarded as a k-complex, where T^nA is viewed in degree $n-1$.

$$0 \leftarrow A \overset{\theta_A}{\leftarrow} TA \overset{\theta_1}{\leftarrow} T^2A \overset{\theta_2}{\leftarrow} \ldots \tag{10}$$

The maps θ_n for $n \geq 1$ are defined in terms of face maps, $\theta_n := \sum_{i=0}^n (-1)^i T^{n-i}\theta_{T^iA}$. Note that naturality of θ impies θ_n is a natural transformation $T^{p+1} \to T^p$.

Remark 34. Note that the complex (10) is nothing more than the complex associated to May's 2-sided bar construction $B_*(T,T,A)$ (See chapter 9 of [14]). If we denote by A_0 the chain complex consisting of A in degree 0 and 0 in higher degrees, then there is a homotopy $h_n : B_n(T,T,A) \to B_{n+1}(T,T,A)$ that establishes a strong deformation retract $B_*(T,T,A) \to A_0$. In fact, the homotopy maps are given by $h_n := h_{T^{n+1}A}$, where h is the natural transformation $U \to UT$ discussed above.

For each $q \geq 0$, if we apply the functor $\mathscr{Y}_q$ to the complex (10), we obtain the sequence below:

$$0 \longleftarrow \mathscr{Y}_qA \xleftarrow{\mathscr{Y}_q\theta_A} \mathscr{Y}_qTA \xleftarrow{\mathscr{Y}_q\theta_1} \mathscr{Y}_qT^2A \xleftarrow{\mathscr{Y}_q\theta_2} \mathscr{Y}_qT^3A \longleftarrow \cdots \tag{11}$$

I claim this sequence is exact. Indeed, by Remark 14, $\mathscr{Y}_qA \cong \bigoplus_{n\geq 0} k\,[N([n] \setminus \Delta S)_{q-1}] \otimes_k A^{\otimes(n+1)}$. For each n, the k-module map h induces a homotopy $h^{\otimes n}$ on each complex,

$$0 \longleftarrow A^{\otimes n} \xleftarrow{\theta_A^{\otimes n}} (TA)^{\otimes n} \xleftarrow{\theta_1^{\otimes n}} (T^2A)^{\otimes n} \longleftarrow \cdots \tag{12}$$

Each $k\,[N([n] \setminus \Delta S)_{q-1}]$ is free as k-module, so tensor products preserve exactness.

Denote by $d_i(A)$ (or d_i, when the context is clear) the i^{th} differential map of $\mathscr{Y}_*A$. Consider the double complex defined by $\mathscr{T}_{p,q} := \mathscr{Y}_q T^{p+1}A$, with vertical differential $(-1)^p d_q$.

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathscr{Y}_2 TA \arrow[d, "d_2"'] & \mathscr{Y}_2 T^2A \arrow[l, "\mathscr{Y}_2\theta_1"'] \arrow[d, "-d_2"'] & \mathscr{Y}_2 T^3A \arrow[l, "\mathscr{Y}_2\theta_2"'] \arrow[d, "d_2"'] & \cdots \arrow[l] \\
\mathscr{Y}_1 TA \arrow[d, "d_1"'] & \mathscr{Y}_1 T^2A \arrow[l, "\mathscr{Y}_1\theta_1"'] \arrow[d, "-d_1"'] & \mathscr{Y}_1 T^3A \arrow[l, "\mathscr{Y}_1\theta_2"'] \arrow[d, "d_1"'] & \cdots \arrow[l] \\
\mathscr{Y}_0 TA & \mathscr{Y}_0 T^2A \arrow[l, "\mathscr{Y}_0\theta_1"'] & \mathscr{Y}_0 T^3A \arrow[l, "\mathscr{Y}_0\theta_2"'] & \cdots \arrow[l]
\end{tikzcd} \tag{13}
$$

Consider a second double complex $\mathscr{A}_{*,*}$ consisting of the complex $\mathscr{Y}_*A$ concentrated in the 0^{th} column.

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathscr{Y}_2 A \arrow[d, "d_2"'] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathscr{Y}_1 A \arrow[d, "d_1"'] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathscr{Y}_0 A & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{14}
$$

Theorem 35. *There is a map of double complexes $\Theta_{*,*} : \mathscr{T}_{*,*} \to \mathscr{A}_{*,*}$ inducing isomorphism in homology, $H_*\left(\mathrm{Tot}(\mathscr{T})\right) \to H_*\left(\mathrm{Tot}(\mathscr{A})\right)$.*

Proof. The map $\Theta_{*,*}$ is defined by $\Theta_{0,q} := \mathscr{Y}_q\theta_A$, and $\Theta_{p,q} = 0$ for positive p. Functoriality of $\mathscr{Y}_*$ ensures that $\mathscr{Y}_*\theta_A$ is a chain map. The isomorphism follows from the exactness of the sequence (11). □

Remark 36. Observe that $H_*\left(\mathrm{Tot}(\mathscr{A}); k\right) = H_*\left(\mathscr{Y}_*A; k\right) = HS_*(A)$. This permits the computation of symmetric homology of any given algebra A in terms of tensor algebras:

Corollary 37. *For an associative, unital k-algebra A, $HS_*(A) \cong H_*\left(\mathrm{Tot}(\mathscr{T}); k\right)$, where $\mathscr{T}_{*,*}$ is the double complex $\{\mathscr{Y}_q T^{p+1}A\}_{p,q\geq 0}$.*

The following lemma shows why it is advantageous to work with tensor algebras.

Lemma 38. *For a unital, associative k-algebra A, there is an isomorphism of k-complexes:*

$$\mathscr{Y}_*TA \cong \bigoplus_{n\geq -1} Y_n(A), \tag{15}$$

where

$$Y_n(A) = \begin{cases} k\left[N(\Delta S)\right], & n = -1 \\ k\left[N([n] \setminus \Delta S)\right] \otimes_{k\Sigma_{n+1}^{\mathrm{op}}} A^{\otimes(n+1)}, & n \geq 0 \end{cases}$$

Moreover, the differential respects the direct sum decomposition.

Proof. Any generator of $\mathscr{Y}_*TA$ has the form $[p] \xrightarrow{\alpha} [q_0] \to \ldots \to [q_n] \otimes u$, where

$$u = \left(\bigotimes_{a\in A_0} a\right) \otimes \left(\bigotimes_{a\in A_1} a\right) \otimes \ldots \otimes \left(\bigotimes_{a\in A_p} a\right),$$

and $A_0, A_1, \ldots, A_p$ are finite ordered lists of elements of A. Indeed, each A_j may be thought of as an element of the set product A^{m_j} for some m_j. If $A_j = \emptyset$, then $m_j = 0$. We use the convention that an empty tensor product is equal to 1_k, and say that the corresponding tensor factor is *trivial*. Now, let $m = (\sum m_j) - 1$. Let $\pi : A^{m_0} \times A^{m_1} \times \ldots \times A^{m_p} \longrightarrow A^{m+1}$ be the evident isomorphism. Let $A_m = \pi(A_0, A_1, \ldots, A_p)$.

Case 1. If u is non-trivial (*i.e.*, $A_m \neq \emptyset$), then construct the element

$$u' = \bigotimes_{a\in A_m} a$$

Next, construct a Δ-morphism $\zeta_u : [m] \to [p]$ as follows: For each j, ζ_u maps the points

$$\sum_{i=0}^{j-1} m_i, \left(\sum_{i=0}^{j-1} m_i\right) + 1, \ldots, \left(\sum_{i=0}^{j} m_i\right) - 1 \;\mapsto\; j$$

Observe, $(\zeta_u)_*(u') = u$. Under ΔS-equivalence, $[p] \xrightarrow{\alpha} [q_0] \to \ldots \to [q_n] \otimes u \approx [m] \xrightarrow{\alpha\zeta_u} [q_0] \to \ldots \to [q_n] \otimes u'$. The choice of u' and ζ_u is well-defined with respect to the ΔS-equivalence up to isomorphism of $[m]$ (an element of $\Sigma_{m+1}^{\mathrm{op}}$). This shows that any such non-trivial element in $\mathscr{Y}_*TA$ may be written uniquely as an element of $k\,[N([m] \setminus \Delta S)] \otimes_{k\Sigma_{m+1}^{\mathrm{op}}} A^{\otimes(m+1)}$.

Case 2. If u is trivial, then $u = 1_k^{\otimes(p+1)}$, and $[p] \xrightarrow{\alpha} [q_0] \to \ldots \to [q_n] \otimes u \approx [q_0] \xrightarrow{\mathrm{id}} [q_0] \to \ldots \to [q_n] \otimes 1_k^{\otimes(q_0+1)}$. This element can be identified uniquely with $[q_0] \to \ldots \to [q_n] \in k\,[N(\Delta S)]$.

Thus, the isomorphism (15) is proven. Note that for $B_*^{sym}TA$, the total number of non-trivial tensor factors is preserved under ΔS morphisms. This shows that the differential respects the direct sum decomposition. $\square$

4. Symmetric Homology of Monoid Algebras

The symmetric homology for the case of a monoid algebra $A = k[M]$ has been studied by Fiedorowicz in [7]. In the most general formulation (Prop. 1.3 of [7]), we have:

Theorem 39. *$HS_*(k[M]) \cong H_*(B(C_\infty, C_1, M); k)$, where C_1 is the little 1-cubes monad, C_∞ is the little ∞-cubes monad, and $B(C_\infty, C_1, M)$ is May's functorial 2-sided bar construction.*

The proof, found in [7], makes use of a variant of the symmetric bar construction:

Definition 40. Let M be a monoid. Define a functor $B_*^{sym}M : \Delta S \to \mathbf{Sets}$ by:

$$B_n^{sym}M := B_*^{sym}M[n] := M^{n+1}, \text{ (set product)}$$
$$B_*^{sym}M(\alpha) : (m_0, \ldots, m_n) \mapsto \alpha(m_0, \ldots, m_n), \qquad \text{for } \alpha \in \mathrm{Mor}\Delta S.$$

where $\alpha : [n] \to [k]$ is represented in tensor notation, and evaluation at $(m_0, \ldots, m_n)$ is as in definition 7.

Definition 41. For a $\mathscr{C}$-set X and $\mathscr{C}^{\mathrm{op}}$-set Y, define the $\mathscr{C}$-equivariant set product:

$$Y \times_{\mathscr{C}} X := \left(\coprod_{C\in\mathrm{Obj}\mathscr{C}} Y(C) \times X(C)\right) / \approx,$$

where the equivalence $\approx$ is generated by the following: For every morphism $f \in \mathrm{Mor}_{\mathscr{C}}(C, D)$, and every $x \in X(C)$ and $y \in Y(D)$, we have $\big(y, f_*(x)\big) \approx \big(f^*(y), x\big)$.

Note that $B_*^{sym}M$ is a ΔS-set, and also a simplicial set, via the chain of functors $\Delta^{\mathrm{op}} \hookrightarrow \Delta C^{\mathrm{op}} \xrightarrow{\cong} \Delta C \hookrightarrow \Delta S$. Let $\mathscr{X}_* := N(- \setminus \Delta S) \times_{\Delta S} B_*^{sym}M$.

Proposition 42. *$\mathscr{X}_*$ is a simplicial set whose homology computes $HS_*(k[M])$.*

Proof. Since M is a k-basis for $k[M]$, $B_*^{sym}M$ acts as a k-basis for $B_*^{sym}k[M]$. Then, observe that

$$k[N(-\setminus \Delta S)\times_{\Delta S} B_*^{sym}M] = k[N(-\setminus\Delta S)]\otimes_{\Delta S} B_*^{sym}k[M].$$

□

If $M = JX_+$ is a free monoid on a generating set X, then $k[M] = T(X)$, the (free) tensor algebra over k on the set X. In this case, we have the following:

Lemma 43.

$$HS_*\left(T(X)\right)\cong H_*\left(\coprod_{n\geq -1}\widetilde{X}_n; k\right),$$

where

$$\widetilde{X}_n = \begin{cases} N(\Delta S), & n=-1\\ N\left([n]\setminus\Delta S\right)\times_{\Sigma_{n+1}^{\mathrm{op}}} X^{n+1}, & n\geq 0\end{cases}$$

Proof. This is essentially a special case of Lemma 38 when the tensor algebra is free, generated by $X = \{x_i \mid i\in\mathscr{I}\}$. □

This proves Thm. 39 in the special case that M is a free monoid. Indeed, if $A = k[JX_+]$, we have

$$HS_*(A)\cong H_*\left(N(\Delta S)\right)\oplus H_*\left(\coprod_{n\geq 0} N\left([n]\setminus\Delta S\right)\times_{\Sigma_{n+1}^{\mathrm{op}}} X^{n+1}\right)$$

Now, the set $\{N\left([n]\setminus\Delta S\right)\}_{n\geq 0}$ has the structure of C_∞-operad, in the sense of May [16], since the augmented chain complex $k\leftarrow k\left[N\left([n]\setminus\Delta S\right)\right]$ is acyclic (Prop. 11), and the action of the symmetric group on $N\left([n]\setminus\Delta S\right)$ is free. Let the associated monad be denoted $\mathscr{N}^{\Delta S}$. Thus we have by definition,

$$\mathscr{N}^{\Delta S}X = \coprod_{n\geq 0} N\left([n]\setminus\Delta S\right)\times_{\Sigma_{n+1}^{\mathrm{op}}} X^{n+1}$$

We obtain a chain of equivalences,

$$\mathscr{N}^{\Delta S}X\simeq B(\mathscr{N}^{\Delta S}, J, JX)\simeq B(C_\infty, J, JX)\simeq B(C_\infty, C_1, JX).$$

The first equivalence arises by Prop. 9.9 of May [14], and the last equivalence arises from a weak homotopy equivalence mentioned in Cor. 6.2 of [14] – the James construction J, is equivalent to the little 1-cubes monad, C_1.

If M is a group, Γ, then Fiedorowicz [7] found:

Theorem 44. $HS_*(k[\Gamma])\cong H_*\left(\Omega\Omega^\infty S^\infty(B\Gamma); k\right)$.

This final formulation shows in particular that HS_* is a non-trivial theory. While it is true that $H_*(\Omega^\infty S^\infty(X)) = H_*(QX)$ is well understood, the same cannot be said of the homology of $\Omega\Omega^\infty S^\infty X$. Indeed, May states that $H_*(QX)$ may be regarded as the free allowable Hopf algebra with conjugation over the Dyer-Lashof algebra and dual of the Steenrod algebra (See [3], preface to chapter 1, and also Lemma 4.10). Cohen and Peterson [4] found the homology of $\Omega\Omega^\infty S^\infty X$, where $X = S^0$, the zero-sphere, but there is little hope of extending this result to arbitrary X using the same methods.

We shall have more to say about $HS_1(k[\Gamma])$ in 11.5.

5. Symmetric Homology Using ΔS_+

In this section, we shall show that replacing ΔS by ΔS_+ in an appropriate way does not affect the computation of HS_*. First, we extend the symmetric bar contruction over ΔS_+.

Definition 45. For an associative, unital algebra, A, over a commutative ground ring k, define a functor $B_*^{sym_+}A : \Delta S_+\to k$-**Mod** by:

$$\begin{cases} B_n^{sym_+}A & := & B_*^{sym}A[n] := A^{\otimes(n+1)}\\ B_{-1}^{sym_+}A & := & k,\end{cases}$$

$$\begin{cases} B_*^{sym_+}A(\alpha) : (a_0\otimes a_1\otimes\ldots\otimes a_n)\mapsto \alpha(a_0, a_1,\ldots, a_n), & \text{for } \alpha\in\mathrm{Mor}\Delta S,\\ B_*^{sym_+}A(\iota_n) : \lambda\mapsto\lambda(1_A\otimes\ldots\otimes 1_A), & (\lambda\in k).\end{cases}$$

Consider the functor $\mathscr{Y}_*^+ : k$-**Alg**$\to k$-**SimpMod** given by

$$\mathscr{Y}_*^+ A \;:=\; k[N(-\setminus \Delta S_+)] \otimes_{\Delta S_+} B_*^{sym_+} A. \tag{16}$$
$$\mathscr{Y}_*^+ f = \mathrm{id} \otimes B_*^{sym_+} f$$

Our goal is to prove the following:

Theorem 46. *For an associative, unital k-algebra A, $HS_*(A) = H_*\left(\mathscr{Y}_*^+ A;\, k\right)$.*

As the preliminary step, we shall prove the theorem in the special case of tensor algebras. We shall need an analogue of Lemma 38 for ΔS_+.

Lemma 47. *For a unital, associative k-algebra A, there is an isomorphism of k-complexes:*

$$\mathscr{Y}_*^+ TA \cong \bigoplus_{n\geq -1} Y_n^+, \tag{17}$$

where

$$Y_n^+ = \begin{cases} k\left[N(\Delta S_+)\right], & n=-1 \\ k\left[N([n]\setminus \Delta S_+)\right] \otimes_{k\Sigma_{n+1}} A^{\otimes(n+1)}, & n \geq 0 \end{cases}$$

Moreover, the differential respects the direct sum decomposition.

Proof. The proof follows verbatim as the proof of Lemma 38, only with ΔS replaced with ΔS_+ throughout. □

Lemma 48. *There is a chain map $J_A : \mathscr{Y}_* A \to \mathscr{Y}_*^+ A$, which is natural in A.*

Proof. First observe that the the inclusion $\Delta S \hookrightarrow \Delta S_+$ induces an inclusion of nerves, $N(-\setminus\Delta S) \hookrightarrow N(-\setminus \Delta S_+)$, which in turn induces the chain map:

$$k\left[N(-\setminus \Delta S)\right] \otimes_{\Delta S} B_*^{sym} A \longrightarrow k\left[N(-\setminus \Delta S_+)\right] \otimes_{\Delta S} B_*^{sym} A. \tag{18}$$

Now, $k\left[N(-\setminus \Delta S_+)\right]$ is both a right ΔS-module and a right ΔS_+-module. Similarly, $B_*^{sym_+}A$ is both a left ΔS-module and a left ΔS_+-module. There is a natural transformation $B_*^{sym} A \to B_*^{sym_+} A$, again induced by inclusion of categories, hence there is a chain map

$$k\left[N(-\setminus \Delta S_+)\right] \otimes_{\Delta S} B_*^{sym} A \longrightarrow k\left[N(-\setminus \Delta S_+)\right] \otimes_{\Delta S} B_*^{sym_+} A. \tag{19}$$

Finally, pass to tensors over ΔS_+:

$$k\left[N(-\setminus \Delta S_+)\right] \otimes_{\Delta S} B_*^{sym_+} A \longrightarrow k\left[N(-\setminus \Delta S_+)\right] \otimes_{\Delta S_+} B_*^{sym_+} A. \tag{20}$$

The composition of maps (18), (19) and (20) defines a chain map $J_A : \mathscr{Y}_* A \to \mathscr{Y}_*^+ A$. It is straightforward to verify that J is natural in A. □

We shall show that J_A induces an isomorphism on homology $H_*\left(\mathscr{Y}_* A;\, k\right) \xrightarrow{\cong} H_*\left(\mathscr{Y}_*^+ A;\, k\right)$ by examining the case of tensor algebras.

Lemma 49. *For a unital, associative k-algebra A, the chain map $J_{TA} : \mathscr{Y}_* TA \longrightarrow \mathscr{Y}_*^+ TA$ is a homotopy equivalence, hence $HS_*(TA) = H_*\left(\mathscr{Y}_*^+ TA;\, k\right)$.*

Proof. There is a commutative square of complexes:

$$\begin{tikzcd} \mathscr{Y}_* TA \arrow[r, "J_{TA}"] & \mathscr{Y}_*^+ TA \\ \bigoplus_{n\geq -1} Y_n \arrow[u, "\cong"] \arrow[r, "\bigoplus_{n\geq -1} j_n"] & \bigoplus_{n\geq -1} Y_n^+ \arrow[u, "\cong"'] \end{tikzcd}$$

The isomorphisms on the left and right follow from Lemmas 38 and 47. The maps j_n are defined by inclusion of categories $\Delta S \hookrightarrow \Delta S_+$:

$$j_{-1} : k\left[N(\Delta S)\right] \to k\left[N(\Delta S_+)\right] \tag{21}$$

$$j_n : k\,[N([n] \setminus \Delta S)] \otimes_{k\Sigma_{n+1}} A^{\otimes(n+1)} \longrightarrow k\,[N([n] \setminus \Delta S_+)] \otimes_{k\Sigma_{n+1}} A^{\otimes(n+1)}, \qquad n \geq 0. \tag{22}$$

Observe that $N(\Delta S_+)$ is contractible, since $[-1] \in \mathrm{Obj}(\Delta S_+)$ is initial. Thus by Lemma 15, the map j_{-1} is a homotopy equivalence on the (-1)-component. Now, for $n \geq 0$, there is equality $N([n]\setminus\Delta S_+) = N([n]\setminus\Delta S)$, since there are no morphisms $[n] \to [-1]$ for $n \geq 0$, so j_n is a homotopy equivalence. Therefore $\bigoplus j_n$, and hence J_{TA}, is a homotopy equivalence. □

Remark 50. Observe, this lemma provides our first major departure from the theory of cyclic homology. The proof above would not work over the categories ΔC and ΔC_+, as $N(\Delta C)$ is not contractible.

Consider a double complex $\mathscr{T}^+_{*,*}(A)$, the analogue of complex (13) for ΔS_+.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathscr{Y}_2^+TA \arrow[d, "d_2"'] & \mathscr{Y}_2^+T^2A \arrow[l, "\mathscr{Y}_2^+\theta_1"'] \arrow[d, "-d_2"'] & \mathscr{Y}_2^+T^3A \arrow[l, "\mathscr{Y}_2^+\theta_2"'] \arrow[d, "d_2"'] & \cdots \arrow[l] \\
\mathscr{Y}_1^+TA \arrow[d, "d_1"'] & \mathscr{Y}_1^+T^2A \arrow[l, "\mathscr{Y}_1^+\theta_1"'] \arrow[d, "-d_1"'] & \mathscr{Y}_1^+T^3A \arrow[l, "\mathscr{Y}_2^+\theta_2"'] \arrow[d, "d_1"'] & \cdots \arrow[l] \\
\mathscr{Y}_0^+TA & \mathscr{Y}_0^+T^2A \arrow[l, "\mathscr{Y}_0^+\theta_1"'] & \mathscr{Y}_0^+T^3A \arrow[l, "\mathscr{Y}_1^+\theta_2"'] & \cdots \arrow[l]
\end{tikzcd} \tag{23}$$

Consider a second double complex, $\mathscr{A}^+_{*,*}(A)$, the analogue of complex (14) for ΔS_+. It consists of the complex $\mathscr{Y}_*^+A$ as the 0^{th} column, and 0 in all positive columns.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathscr{Y}_2^+A \arrow[d, "d_2"'] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathscr{Y}_1^+A \arrow[d, "d_1"'] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathscr{Y}_0^+A & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{24}$$

We may think of each double complex construction, $\mathscr{A}_{*,*}$, $\mathscr{T}_{*,*}$, $\mathscr{A}^+_{*,*}$ and $\mathscr{T}^+_{*,*}$, as a functor from k-**Alg** to the category of double k-complexes. Each functor takes unital morphisms of algebras to maps of double complexes in the obvious way – for example if $f : A \to B$, then the induced map $\mathscr{T}_{*,*}(A) \to \mathscr{T}_{*,*}(B)$ is defined on the (p, q)-component by the map $\mathscr{Y}_qT^{p+1}f$. The induced map commutes with vertical differentials of $\mathscr{A}_{*,*}$ and $\mathscr{T}_{*,*}$ (resp., $\mathscr{A}^+_{*,*}$ and $\mathscr{T}^+_{*,*}$) by naturality of $\mathscr{Y}_*$ (resp. $\mathscr{Y}_*^+$), and it commutes with the horizontal differentials of $\mathscr{T}_{*,*}$ and $\mathscr{T}^+_{*,*}$ by naturality of θ_n. The map J induces a natural transformation $J_{*,*} : \mathscr{A}_{*,*} \to \mathscr{A}^+_{*,*}$, defined by

$$J_{p,*}(A) = \begin{cases} J_A : \mathscr{Y}_*A \to \mathscr{Y}_*^+A, & p = 0 \\ 0, & p > 0 \end{cases}$$

Define a map of bigraded modules, $K_{*,*}(A) : \mathscr{T}_{*,*}(A) \to \mathscr{T}^+_{*,*}(A)$ by: $K_{p,*}(A) = J_{T^{p+1}A} : \mathscr{Y}_*T^{p+1}A \to \mathscr{Y}_*^+T^{p+1}A$. Now, $K_{*,*}(A)$ commutes with the vertical differentials because each $J_{T^{p+1}A}$ is a chain map.

$K_{*,*}(A)$ commutes with the horizontal differentials because of naturality of J. Finally, $K_{*,*}$ defines a natural transformation $\mathscr{T}_{*,*} \to \mathscr{T}^{+}_{*,*}$, again by naturality of J.

$$
\begin{tikzcd}
A \arrow[d, "f"'] & \mathscr{Y}_q T^{p+1}A \arrow[r, "K_{p,q}(A)", "=J_{T^{p+1}A}"'] \arrow[d, "\mathscr{Y}_q T^{p+1}f"'] & \mathscr{Y}_q^{+} T^{p+1}A \arrow[d, "\mathscr{Y}_q^{+} T^{p+1}f"] \\
B & \mathscr{Y}_q T^{p+1}A \arrow[r, "K_{p,q}(B)", "=J_{T^{p+1}B}"'] & \mathscr{Y}_q^{+} T^{p+1}A
\end{tikzcd}
$$

Recall by Thm 35, there is a map of double complexes, $\Theta_{*,*}(A) : \mathscr{T}_{*,*}(A) \longrightarrow \mathscr{A}_{*,*}(A)$ inducing an isomorphism in homology of the total complexes. We shall need the analogous statement for the double complexes $\mathscr{T}^{+}_{*,*}$ and $\mathscr{A}^{+}_{*,*}$.

Theorem 51. *For any unital associative algebra, A, there is a map of double complexes, $\Theta^{+}_{*,*}(A) : \mathscr{T}^{+}_{*,*}(A) \to \mathscr{A}^{+}_{*,*}(A)$ inducing isomorphism in homology $H_*\left(\mathrm{Tot}(\mathscr{T}^{+}(A));\, k\right) \to H_*\left(\mathrm{Tot}(\mathscr{A}^{+}(A));\, k\right)$. Moreover, $\Theta^{+}_{*,*}$ provides a natural transformation $\mathscr{T}^{+}_{*,*} \to \mathscr{A}^{+}_{*,*}$.*

Proof. The map $\Theta^{+}_{*,*}(A)$ is defined as:

$$\Theta^{+}_{p,q}(A) := \begin{cases} \mathscr{Y}_q^{+}\theta_A, & p = 0 \\ 0, & p > 0 \end{cases}$$

This map is a map of double complexes by functoriality of $\mathscr{Y}^{+}_{*}$, and the isomorphism follows from the exactness of the sequence (11). Naturality of $\Theta_{*,*}$ follows from naturality of θ. $\square$

Lemma 52. *The following diagram of functors and transformations is commutative.*

$$
\begin{tikzcd}
\mathscr{T}_{*,*} \arrow[r, "\Theta_{*,*}"] \arrow[d, "K_{*,*}"'] & \mathscr{A}_{*,*} \arrow[d, "J_{*,*}"] \\
\mathscr{T}^{+}_{*,*} \arrow[r, "\Theta^{+}_{*,*}"] & \mathscr{A}^{+}_{*,*}
\end{tikzcd}
\tag{25}
$$

Proof. It suffices to fix an algebra A and examine only the (p,q)-components. Note, if $p > 0$, then the right hand side of the diagram is trivial, so we may assume $p = 0$.

$$
\begin{tikzcd}
\mathscr{Y}_q TA \arrow[r, "\mathscr{Y}_q\theta_A"] \arrow[d, "(J_{TA})_q"'] & \mathscr{Y}_q A \arrow[d, "(J_A)_q"] \\
\mathscr{Y}_q^{+} TA \arrow[r, "\mathscr{Y}_q^{+}\theta_A"] & \mathscr{Y}_q^{+} A
\end{tikzcd}
\tag{26}
$$

This diagram commutes by naturality of J. $\square$

To any double complex $\mathscr{B}_{*,*}$ over k, we may associate two spectral sequences: $(E_I\mathscr{B})_{*,*}$, obtained by first taking vertical homology, then horizontal; and $(E_{II}\mathscr{B})_{*,*}$, obtained by first taking horizontal homology, then vertical. In the case that $\mathscr{B}_{*,*}$ lies entirely within the first quadrant, both spectral sequences converge to $H_*\left(\mathrm{Tot}(\mathscr{B});\, k\right)$ (See [18], Section 2.4). Maps of double complexes induce maps of spectral sequences, E_I and E_{II}, respectively.

Fix the algebra A, and consider the commutative diagram of spectral sequences induced by diagram (25). The induced maps will be indicated by an overline, and explicit mention of the algebra A is suppressed for

brevity of notation.

$$
\begin{tikzcd}
E_{II}\mathscr{T} \arrow[r, "\overline{\Theta}"] \arrow[d, "\overline{K}"'] & E_{II}\mathscr{A} \arrow[d, "\overline{J}"] \\
E_{II}\mathscr{T}^+ \arrow[r, "\overline{\Theta^+}"] & E_{II}\mathscr{A}^+
\end{tikzcd}
\tag{27}
$$

Now, by Thm. 35 and Thm. 51, we know that $\Theta_{*,*}$ and $\Theta^+_{*,*}$ induce isomorphisms on total homology, so $\overline{\Theta}$ and $\overline{\Theta^+}$ also induce isomorphisms on the limit term of the spectral sequences. In fact, both $\overline{\Theta}^r$ and $\overline{\Theta^+}^r$ are isomorphisms $(E_{II}\mathscr{T})^r \to (E_{II}\mathscr{A})^r$ for $r \geq 1$. This is because taking horizontal homology of $\mathscr{T}_{*,*}$ (resp. $\mathscr{T}^+_{*,*}$) kills all components in positive columns, leaving only the 0^{th} column, which is chain-isomorphic to the 0^{th} column of $\mathscr{A}_{*,*}$ (resp. $\mathscr{A}^+_{*,*}$).

Consider a second diagram of spectral sequences, with induced maps indicated by a hat.

$$
\begin{tikzcd}
E_{I}\mathscr{T} \arrow[r, "\widehat{\Theta}"] \arrow[d, "\widehat{K}"'] & E_{I}\mathscr{A} \arrow[d, "\widehat{J}"] \\
E_{I}\mathscr{T}^+ \arrow[r, "\widehat{\Theta^+}"] & E_{I}\mathscr{A}^+
\end{tikzcd}
\tag{28}
$$

Now the map $\widehat{K}$ induces an isomorphism on the limit terms of the sequences $E_I\mathscr{T}$ and $E_I\mathscr{T}^+$ as a result of Lemma 49. As before, $\widehat{K}^r$ is an isomorphism for $r \geq 1$.

Now, since $H_*(\mathrm{Tot}(\mathscr{A}); k) = H_*(\mathscr{Y}_*A; k)$ and $H_*(\mathrm{Tot}(\mathscr{A}^+); k) = H_*(\mathscr{Y}^+_*A; k)$, we can put together a chain of isomorphisms

$$
\begin{aligned}
H_*(\mathscr{Y}_*A; k) \cong (E_{II}\mathscr{A})^\infty_* \xleftarrow[\cong]{\overline{\Theta}^\infty} (E_{II}\mathscr{T})^\infty_* \cong (E_I\mathscr{T})^\infty_* \xrightarrow[\cong]{\widehat{K}^\infty_*} \left(E_I\mathscr{T}^+\right)^\infty_* \\
\cong \left(E_{II}\mathscr{T}^+\right)^\infty_* \xrightarrow[\cong]{(\overline{\Theta^+})^\infty_*} \left(E_{II}\mathscr{A}^+\right)^\infty_* \cong H_*\left(\mathscr{Y}^+_*A; k\right)
\end{aligned}
\tag{29}
$$

Commutativity of Diagram (25) ensures that the composition of maps in Diagram (29) is the map induced by J_A, hence proving:

Theorem 53. *For a unital, associative k-algebra A, the chain map $J_A : \mathscr{Y}_*A \to \mathscr{Y}^+_*A$ induces an isomorphism on homology, $H_*(\mathscr{Y}_*A; k) \xrightarrow{\cong} H_*(\mathscr{Y}^+_*A; k)$.*

As a direct consequence, $HS_*(A) \cong H_*(\mathscr{Y}^+_*A; k)$, proving Thm. 46.

6. The Category EpiΔS and a Smaller Resolution

The complex (1) is an extremely large and unwieldy for computation. Fortunately, when the algebra A is equipped with an augmentation, $\epsilon : A \to k$, complex (1) is homotopic to a much smaller subcomplex.

6.1. **Basic and Reduced Tensors.** Recall, if A has an augmentation ϵ, then there is an augmentation ideal I and the sequence $0 \to I \to A \xrightarrow{\epsilon} k \to 0$ is split exact as k-modules. So $A \cong I \oplus k$, and every $x \in A$ can be decomposed uniquely as $x = a + \lambda$ for some $a \in I$, $\lambda \in k$.

Definition 54. Define $B_{-1,\emptyset}A = k$. For $n \geq 0$, if $J \subseteq [n]$, define

$$
B_{n,J}A := B^J_0 \otimes B^J_1 \otimes \ldots \otimes B^J_n, \quad \text{where } B^J_j = \begin{cases} I & \text{if } j \in J \\ k & \text{if } j \notin J \end{cases}
$$

Lemma 55. *For each $n \geq -1$, there is a direct sum decomposition of k-modules $B^{sym_+}_n A \cong \bigoplus_{J \subseteq [n]} B_{n,J}A$.*

Proof. For $n = -1$, $B^{sym_+}_{-1}A = k = B_{-1,\emptyset}A$. For $n \geq 0$, $B^{sym_+}_n A = (I \oplus k)^{\otimes(n+1)} \cong \bigoplus_{J \subseteq [n]} B_{n,J}A$. $\square$

Definition 56. A *basic tensor* is any tensor $w_0 \otimes w_1 \otimes \ldots \otimes w_n$, where each w_j is in I or is equal to the unit of A. Call a tensor factor w_j *trivial* if it is the unit of A. If all factors of a basic tensor are trivial, then the tensor is called *trivial*, and if no factors are trivial, the tensor is called *reduced*.

It will become convenient to include the object $[-1] = \emptyset$ in Δ. Denote the enlarged category by Δ_+. For a basic tensor $Y \in B_n^{sym_+}A$, we shall define a map $\delta_Y \in \mathrm{Mor}\Delta_+$ as follows: If Y is trivial, let $\delta_Y = \iota_n$. Otherwise, Y has $\overline{n}+1$ non-trivial factors for some $\overline{n} \geq 0$. Define $\delta_Y : [\overline{n}] \to [n]$ to be the unique injective map that sends each point $0, 1, \ldots, \overline{n}$ to a point $p \in [n]$ such that Y is non-trivial at factor p. Let $\overline{Y}$ be the tensor obtained from Y by omitting all trivial factors if such exist, or $\overline{Y} := 1_k$ if Y is trivial. Note, $\overline{Y}$ is the unique basic tensor such that $(\delta_Y)_*(\overline{Y}) = Y$.

Proposition 57. *Any chain* $[q] \to [q_0] \to \ldots \to [q_n] \otimes Y \;\in\; k[N(- \setminus \Delta S_+)] \otimes_{\Delta S_+} B_*^{sym_+}A$, *where* Y *is a basic tensor, is equivalent to a chain* $[\overline{q}] \to [q_0] \to \ldots \to [q_n] \otimes \overline{Y}$, *where either* $\overline{Y}$ *is reduced or* $\overline{Y} = 1_k$ *and* $\overline{q} = -1$.

Proof. Let δ_Y and $\overline{Y}$ be defined as above, and let $[\overline{q}]$ be the source of δ_Y. Then $Y = (\delta_Y)_*(\overline{Y})$, and

$$[q] \stackrel{\phi}{\to} [q_0] \to \ldots \to [q_n] \otimes Y \;\approx\; [\overline{q}] \stackrel{\phi\delta_Y}{\to} [q_0] \to \ldots \to [q_n] \otimes \overline{Y}.$$

□

6.2. **Reducing to Epimorphisms.** We now turn our attention to the morphisms in the chains. Our goal is to reduce to those chains that involve only epimorphisms.

Definition 58. Let $\mathscr{C}$ be a category. The category $\mathrm{Epi}\mathscr{C}$ (resp. $\mathrm{Mono}\mathscr{C}$) is the subcategory of $\mathscr{C}$ consisting of the same objects as $\mathscr{C}$ and only those morphisms $f \in \mathrm{Mor}\mathscr{C}$ that are epic (resp. monic).

The set of morphisms of $\mathrm{Epi}\mathscr{C}$ from X to Y may be denoted $\mathrm{Epi}_{\mathscr{C}}(X, Y)$. Similarly, the set of morphisms of $\mathrm{Mono}\mathscr{C}$ from X to Y may be denoted $\mathrm{Mono}_{\mathscr{C}}(X, Y)$.

Remark 59. A morphism $\alpha = (\phi, g) \in \mathrm{Mor}\Delta S_+$ is epic (resp. monic) if and only if ϕ is epic (resp. monic) as morphism in Δ_+.

Proposition 60. *Any morphism* $\alpha \in \mathrm{Mor}\Delta S_+$ *decomposes uniquely as* $(\eta, \mathrm{id}) \circ \gamma$, *where* $\gamma \in \mathrm{Mor}(\mathrm{Epi}\Delta S_+)$ *and* $\eta \in \mathrm{Mor}(\mathrm{Mono}\Delta_+)$.

Proof. Suppose α has source $[-1]$ and target $[n]$. Then $\alpha = \iota_n$ is the only possibility, and this decomposes as $\iota_n \circ \mathrm{id}_{[-1]}$. Now suppose the source of α is $[p]$ for some $p \geq 0$. Write $\alpha = (\phi, g)$, with $\phi \in \mathrm{Mor}\Delta$ and $g \in \Sigma_{p+1}^{\mathrm{op}}$. We shall decompose ϕ as follows: For $\phi : [p] \to [r]$, suppose ϕ hits $q+1$ distinct points in $[r]$. Then $\pi : [p] \to [q]$ is induced by ϕ by maintaining the order of the points hit. η is the obvious order-preserving monomorphism $[q] \to [r]$ so that $\eta\pi = \phi$ as morphisms in Δ. To get the required decomposition in ΔS, use: $\alpha = (\eta, \mathrm{id}) \circ (\pi, g)$.

Now, if $(\xi, \mathrm{id}) \circ (\psi, h)$ is also a decomposition of α, with ξ monic and ψ epic, then $(\xi, \mathrm{id}) \circ (\psi, h) = (\eta, \mathrm{id}) \circ (\pi, g)$ implies $(\xi\psi, g^{-1}h) = (\eta\pi, \mathrm{id})$, proving $g = h$. Uniqueness will then follow from uniqueness of such decompositions entirely within the category Δ. The latter follows from Theorem B.2 of [12], since any monomorphism (resp. epimorphism) of Δ can be factored uniquely as compositions of δ_i (resp. σ_i). □

This decompisition will be written: $[p] \to [r] \;=\; [p] \twoheadrightarrow \mathrm{im}([p] \to [r]) \hookrightarrow [r]$.

Proposition 61. *The epimorphism construction is a functor* $\mathscr{E}_p : [p] \setminus \Delta S_+ \to [p] \setminus \mathrm{Epi}\Delta S_+$.

Proof. Fix $p \geq -1$. If β is an object of $[p] \setminus \Delta S_+$ (*i.e.* a morphism $[p] \to [r_1]$), then $\mathscr{E}(\beta)$ is the epimorphism $[p] \twoheadrightarrow \mathrm{im}([p] \to [r])$. If $[p] \stackrel{\beta}{\to} [r_1] \stackrel{\alpha}{\to} [r_2]$, then there is an induced map $\mathrm{im}([p] \to [r_1]) \stackrel{\overline{\alpha}}{\to} \mathrm{im}([p] \to [r_2])$ making the diagram commute:

$$\begin{tikzcd}
{[r_1]} \arrow[rr, "\alpha"] & & {[r_2]} \\
& {[p]} \arrow[ul, "\beta"'] \arrow[ur, "\alpha\beta"] \arrow[dl, two heads, "\pi_1"] \arrow[dr, two heads, "\pi_2"'] & \\
\mathrm{im}([p] \to [r_1]) \arrow[uu, hook, "\eta_1"] \arrow[rr, two heads, "\overline{\alpha}"'] & & \mathrm{im}([p] \to [r_2]) \arrow[uu, hook', "\eta_2"']
\end{tikzcd} \tag{30}$$

$\overline{\alpha}$ is the epimorphism induced from the map $\alpha\eta_1$. Furthermore, for morphisms $[p] \to [r_1] \xrightarrow{\alpha_1} [r_2] \xrightarrow{\alpha_2} [r_3]$, we have: $\overline{\alpha_2\alpha_1} = \overline{\alpha_2} \circ \overline{\alpha_1}$. □

Remark 62. Note that if $\alpha : [p] \to [r]$ is an epimorphism of ΔS_+, then $\mathscr{E}(\alpha) = \alpha$.

Define a variant of the symmetric bar construction:

Definition 63. $B_*^{sym_+}I : \mathrm{Epi}\Delta S_+ \to k$-**Mod** is the functor defined by:

$$\begin{cases} B_n^{sym_+}I & := & I^{\otimes(n+1)}, \quad n \geq 0, \\ B_{-1}^{sym_+}I & := & k, \end{cases}$$

$$B_*^{sym_+}I(\alpha) : (a_0 \otimes a_1 \otimes \ldots \otimes a_n) \mapsto \alpha(a_0, \ldots, a_n), \text{ for } \alpha \in \mathrm{Mor}(\mathrm{Epi}\Delta S_+)$$

This definition makes sense since I is an ideal, and α is required to be epimorphic. Note, the simple tensors $w_0 \otimes \ldots \otimes w_n$ in $B_n^{sym_+}I$ are by definition reduced. Consider the simplicial k-module:

$$\mathscr{Y}_*^{epi}A := k[N(- \setminus \mathrm{Epi}\Delta S_+)] \otimes_{\mathrm{Epi}\Delta S_+} B_*^{sym_+}I \tag{31}$$

There is an obvious inclusion, $f : \mathscr{Y}_*^{epi}A \longrightarrow \mathscr{Y}_*^{+}A$. Define a chain map g in the opposite direction as follows. First, by Prop. 57 and observations above, we only need to define g on the chains $u = [q] \to [q_0] \to \ldots \to [q_n] \otimes Y$ where Y is reduced (or $Y = 1_k$). In this case, define component maps $g(q) := N(\mathscr{E}_q) \otimes \mathrm{id}$. A priori, this definition is well-defined only when the tensor product is over k. We would like to assemble the maps $g(q)$ into a chain map g. In order to do this, we must show that the maps are compatible under ΔS_+-equivalence.

Suppose $v = [p] \xrightarrow{\phi\psi} [p_0] \to \ldots \to [p_n] \otimes Z$ and $v' = [q] \xrightarrow{\phi} [p_0] \to \ldots \to [p_n] \otimes \psi_*(Z)$, where $\psi \in \mathrm{Mor}_{\Delta S_+}([p], [q])$. If Z is a basic tensor, then so is $\psi_*(Z)$. In order to apply g to v or v', each must first be put into a reduced form.

Case 1 Suppose Z is trivial. Then v and v' both reduce to $[-1] \to [p_0] \to \ldots \to [p_n] \otimes 1$, hence $g(v) = g(v')$.

Case 2 Suppose Z is non-trivial. For the sake of clean notation, let $W = \psi_*(Z)$. Construct δ_Z, $\overline{Z}$, δ_W and $\overline{W}$ such that $Z = (\delta_Z)_*(\overline{Z})$ and $W = (\delta_W)_*(\overline{W})$, as in Prop. 57, and reduce both chains:

$$\begin{tikzcd} {[p] \xrightarrow{\phi\psi} [p_0] \to \ldots \to [p_n] \otimes Z} \arrow[d, "reduce"', "\approx"] \arrow[r, phantom, "\approx"] & {[q] \xrightarrow{\phi} [p_0] \to \ldots \to [p_n] \otimes W} \arrow[d, "reduce"', "\approx"] \\ {[\overline{p}] \xrightarrow{\phi\psi\delta_Z} [p_0] \to \ldots \to [p_n] \otimes \overline{Z}} & {[\overline{q}] \xrightarrow{\phi\delta_W} [p_0] \to \ldots \to [p_n] \otimes \overline{W}} \end{tikzcd} \tag{32}$$

Observe that number of distinct points hit by $\psi\delta_Z$ is exactly $\overline{q}+1$; indeed, $W = \psi_*(Z)$ has $\overline{q}+1$ non-trivial factors. Thus, $[\overline{q}] = \mathrm{im}([\overline{p}] \to [q])$. Now, Prop. 60 implies that there is precisely one ΔS-epimorphism $\gamma : [\overline{p}] \to [\overline{q}]$ making Diagram (33) commute.

$$\begin{tikzcd} {[p]} \arrow[r, "\psi"] & {[q]} & \\ {[\overline{p}]} \arrow[u, "\delta_Z"] \arrow[ur, "\psi\delta_Z"] \arrow[r] \arrow[drr, two heads, "\gamma"'] & {[\overline{q}]} \arrow[u, "\delta_W"'] \arrow[dr, equal] & \\ & & \mathrm{im}([\overline{p}] \to [q]) \arrow[ul, hook, bend right] \end{tikzcd} \tag{33}$$

That is to say, there exists an epimorphism γ such that $\gamma_*(\overline{Z}) = \overline{W}$ and $\psi\delta_Z = \delta_W\gamma$. So we may replace the first morphism of the chain in the lower left of Diagram (32) with $\phi\delta_W\gamma$. Then when we apply g to the

chain, the first morphism becomes $\mathscr{E}_{\overline{p}}(\phi\delta_W\gamma) = \mathscr{E}_{\overline{p}}(\phi\delta_W) \circ \gamma$, since γ is epic. Let $\pi := \mathscr{E}_{\overline{p}}(\phi\delta_W)$. Then the result of applying g to each side Diagram (32) is shown below:

(34)

$$\begin{array}{ccc} [\overline{p}] \xrightarrow{\phi\psi\delta_Z} [p_0] \to \ldots \to [p_n] \otimes \overline{Z} & & [\overline{q}] \xrightarrow{\phi\delta_W} [p_0] \to \ldots \to [p_n] \otimes \overline{W} \\ \Big\downarrow{\scriptstyle g(\overline{p})} & & \Big\downarrow{\scriptstyle g(\overline{q})} \\ [\overline{p}] \xrightarrow{\pi\gamma} \mathrm{im}([\overline{p}] \to [p_0]) \to \ldots \to \mathrm{im}([\overline{p}] \to [p_n]) \otimes \overline{Z} & & [\overline{q}] \xrightarrow{\pi} \mathrm{im}([\overline{q}] \to [p_0]) \to \ldots \to \mathrm{im}([\overline{q}] \to [p_n]) \otimes \overline{W} \end{array}$$

Observe that there is equality of objects and morphisms, $\mathrm{im}([\overline{p}] \to [p_0]) \to \ldots \to \mathrm{im}([\overline{p}] \to [p_n]) = \mathrm{im}([\overline{q}] \to [p_0]) \to \ldots \to \mathrm{im}([\overline{q}] \to [p_n])$. Finally, since γ is epic, the $\mathrm{Epi}\Delta S_+$-equivalence allows us to identify $g(v) \approx g(v')$. This shows that g is well-defined.

Now clearly $gf = \mathrm{id}$.

Proposition 64. $fg \simeq \mathrm{id}$.

Proof. In what follows, we assume Y is a basic tensor in $B_q^{sym_+}I$. Define maps $h_j^{(n)}$ as follows:

$$h_j^{(n)}\left([q] \to [q_0] \to \ldots \to [q_n] \otimes Y\right) :=$$
$$[q] \twoheadrightarrow \mathrm{im}([q] \to [q_0]) \twoheadrightarrow \ldots \twoheadrightarrow \mathrm{im}([q] \to [q_j]) \hookrightarrow [q_j] \to \ldots \to [q_n] \otimes Y.$$

$h_j^{(n)}$ is well-defined by the functorial properties of the epimorphism construction, and a routine, but tedious, verification shows that h defines a presimplicial homotopy from fg to id. ☐

Proposition 65. *If A has augmentation ideal I, then*

$$HS_*(A) = H_*\left(\mathscr{Y}_*^{epi}A;\, k\right) = H_*\left(k[N(- \setminus \mathrm{Epi}\Delta S_+)] \otimes_{\mathrm{Epi}\Delta S_+} B_*^{sym_+}I;\, k\right).$$

Proof. The complex (31) has been shown to be chain homotopy equivalent to the complex $\mathscr{Y}_*^+A$, which by Thm. 46, computes $HS_*(A)$. ☐

Remark 66. The condition that A have an augmentation ideal may be lifted (as Richter conjectures), if it can be shown that $N(\mathrm{Epi}\Delta S)$ is contractible. As partial progress along these lines, it can be shown that $N(\mathrm{Epi}\Delta S)$ is simply-connected.

7. A spectral sequence for $HS_*(A)$

Fix a unital associative algebra A over commutative ground ring k, and assume A has an augmentation, with augmentation ideal I. Let $\mathscr{Y}_* = \mathscr{Y}_*^{epi}A$ be the complex (31). We shall find a useful spectral sequence that computes $HS_*(A)$ based on a filtration of $\mathscr{Y}_*$.

7.1. **Filtering by Number of Strict Epimorphisms.** Appealing to Remark 14, we shall represent a q-chain of $\mathscr{Y}_*$ as $[m_0] \to [m_1] \to \ldots \to [m_q] \otimes Y$, where the tensor product is over k. Consider a filtration of $\mathscr{Y}_*$ by number of strict epimorphisms, or *jumps* in such chains: $\mathscr{F}_p\mathscr{Y}_q$ is generated by the chains $[m_0] \to [m_1] \to \ldots \to [m_q] \otimes Y$, where $m_{i-1} > m_i$ for no more than p distinct values of i. The face maps of $\mathscr{Y}_*$ only compose or delete morphisms, so $\mathscr{F}_p$ is compatible with the differential of $\mathscr{Y}_*$. The filtration quotients are easily described: $E^0_{p,q} := \mathscr{F}_p\mathscr{Y}_q/\mathscr{F}_{p-1}\mathscr{Y}_q$ is generated by chains $[m_0] \to [m_1] \to \ldots \to [m_q] \otimes Y$, where $m_{i-1} > m_i$ for exactly p distinct values of i. Consider the spectral sequence with $E^1_{p,q} = H_{p+q}(E^0_{p,*})$.

Lemma 67. *There are chain maps (one for each p):*

$$E^0_{p,*} \to \bigoplus_{m_0 > \ldots > m_p} \left(I^{\otimes(m_0+1)} \otimes k\Big[\prod_{i=1}^{p} \mathrm{Epi}_{\Delta_+}\big([m_{i-1}], [m_i]\big)\Big] \otimes_{k\Sigma_{m_p+1}} E_*\Sigma_{m_p+1}\right),$$

inducing isomorphisms in homology:

$$E^1_{p,q} \cong \bigoplus_{m_0 > \ldots > m_p} H_q\left(\Sigma^{\mathrm{op}}_{m_p+1}\,;\, I^{\otimes(m_0+1)} \otimes k\Big[\prod_{i=1}^{p} \mathrm{Epi}_{\Delta_+}\big([m_{i-1}], [m_i]\big)\Big]\right).$$

We use the convention that $I^{\otimes 0} = k$, and $\Sigma_0 \cong 1$, the trivial group.

Remark 68. Here, we are using the resolution E_*G of k by free kG-modules, $E_nG = k\big[\prod^{n+1} G\big]$, with G-action $g.(g_0, g_1, \ldots, g_n) = (gg_0, g_1, \ldots, g_n)$, and differential,

$$\partial^{EG}(g_0, g_1, \ldots, g_n) = \left[\sum_{i=0}^{n-1}(g_0, \ldots, g_ig_{i+1}, \ldots, g_n)\right] + (g_0, g_1, \ldots, g_{n-1}).$$

7.2. **Proof of Lemma 67.** The proof will be broken down into a number of steps. Begin by defining two related chain complexes $\mathscr{B}_*^{(m_0,\ldots,m_p)}$ and $\mathscr{M}_*^{(m_0,\ldots,m_p)}$.

For $m_0 > m_1 > \ldots > m_p$, define:

$$\mathscr{B}_*^{(m_0,\ldots,m_p)} := k\Big[\coprod\Big([m_0] \xrightarrow{\cong} \ldots \xrightarrow{\cong} [m_0] \twoheadrightarrow [m_1] \xrightarrow{\cong} \ldots \xrightarrow{\cong} [m_{p-1}] \twoheadrightarrow [m_p]\Big] \otimes I^{\otimes(m_0+1)},$$

where the coproduct extends over all such chains that begin with 0 or more isomorphisms of $[m_0]$, followed by a strict epimorphism $[m_0] \twoheadrightarrow [m_1]$, followed by 0 or more isomorphisms of $[m_1]$, followed by a strict epimorphism $[m_1] \twoheadrightarrow [m_2]$, etc., and the last morphism must be a strict epimorphism $[m_{p-1}] \twoheadrightarrow [m_p]$. $\mathscr{B}_*^{(m_0,\ldots,m_p)}$ is a subcomplex of $E^0_{p,*}$ with the same induced differential, which we will denote by ∂^B.

Denote by $\mathscr{M}_*^{(m_0,\ldots,m_p)}$, the chain complex consisting of 0 in degrees different from p, and

$$\mathscr{M}_p^{(m_0,\ldots,m_p)} \;:=\; I^{\otimes(m_0+1)} \otimes k\Big[\prod_{i=1}^{p} \mathrm{Epi}_{\Delta_+}\big([m_{i-1}], [m_i]\big)\Big].$$

This is the coefficient group that shows up in Lemma 67. Its differential is trivial. Now, $\mathscr{B}_p^{(m_0,\ldots,m_p)}$ is generated by elements of the form $[m_0] \twoheadrightarrow [m_1] \twoheadrightarrow \ldots \twoheadrightarrow [m_p] \otimes Y$, where the chain consists entirely of strict epimorphisms of ΔS_+. Observe that

$$\mathscr{B}_p^{(m_0,\ldots,m_p)} = k\big[\mathrm{Epi}_{\Delta S_+}([m_0], [m_1])\big] \otimes \ldots \otimes k\big[\mathrm{Epi}_{\Delta S_+}([m_{p-1}], [m_p])\big] \otimes I^{\otimes(m_0+1)}.$$

For convenience, the factor $I^{\otimes(m_0+1)}$ should be put on the left:

$$\mathscr{B}_p^{(m_0,\ldots,m_p)} \cong I^{\otimes(m_0+1)} \otimes k\big[\mathrm{Epi}_{\Delta S_+}([m_0], [m_1])\big] \otimes \ldots \otimes k\big[\mathrm{Epi}_{\Delta S_+}([m_{p-1}], [m_p])\big] \tag{35}$$

Now, each $k\big[\mathrm{Epi}_{\Delta S_+}([m], [n])\big]$ is a $(k\Sigma_{m+1})$-(Σ_{n+1})-bimodule via the action of symmetric group elements as automorphisms of $[m]$ and $[n]$. Explicitly, for an element (ψ, g) of $\mathrm{Epi}_{\Delta S_+}([m], [n])$, and group elements $\tau \in \Sigma_{m+1}$ and $\sigma \in \Sigma_{n+1}$, view τ as an automorphism $t \in \Sigma_{m+1}^{\mathrm{op}}$ and σ as an automorphism $s \in \Sigma_{n+1}^{\mathrm{op}}$. Then $(\psi, g).\sigma \;:=\; (\mathrm{id}, s) \circ (\psi, g) \;=\; (\psi^s, gs^\psi)$, and $\tau.(\psi, g) \;:=\; (\psi, g) \circ (\mathrm{id}, t) \;=\; (\psi, tg)$. Moreover, view $I^{\otimes(m_0+1)}$ as a right $(k\Sigma_{m_0+1})$-module via the identification $I^{\otimes(m_0+1)} = B_{m_0}^{sym_+}I$. With this in mind, (35) becomes a (left) $(k\Sigma^{\mathrm{op}}_{m_p+1})$-module, where the action is the right action of $k\Sigma_{m_p+1}$ on the last tensor factor.

I claim that there is a k-module isomorphism,

$$\mathscr{M}_p^{(m_0,\ldots,m_p)} \;\cong\; I^{\otimes(m_0+1)} \otimes_{kG_0} k\big[\mathrm{Epi}_{\Delta S_+}([m_0], [m_1])\big] \otimes_{kG_1} \ldots \otimes_{kG_{p-1}} k\big[\mathrm{Epi}_{\Delta S_+}([m_{p-1}], [m_p])\big], \tag{36}$$

where G_i is the group Σ_{m_i+1}. The isormorphism follows from the following observation. Any element $Y \otimes (\psi_1, g_1) \otimes \ldots \otimes (\psi_p, g_p)$ in the module on the right in (36) is equivalent to one in which all g_i are identities by first writing $(\psi_p, g_p) = g_p.(\psi_p, \mathrm{id})$ then commuting g_p over the tensor to its left and iterating this process to the leftmost tensor factor. Thus, we may write the element uniquely as $Z \otimes \phi_1 \otimes \ldots \otimes \phi_p$, where all tensors are now over k, and all morphisms are in $\mathrm{Epi}\Delta S_+$, that is, $Z \otimes \phi_1 \otimes \ldots \otimes \phi_p \in \mathscr{M}_p^{(m_0,\ldots,m_p)}$.

Proposition 69. *There is a $\Sigma^{\mathrm{op}}_{m_p+1}$-equivariant chain map, $\gamma_* : \mathscr{B}_*^{(m_0,\ldots,m_p)} \to \mathscr{M}_*^{(m_0,\ldots,m_p)}$, inducing an isomorphism on homology, $H_*\left(\mathscr{B}_*^{(m_0,\ldots,m_p)}\right) \overset{\gamma_*}{\cong} H_*\left(\mathscr{M}_*^{(m_0,\ldots,m_p)}\right)$.*

Proof. γ_* is defined to be 0 in degrees different from p. In degree p, use the isomorphisms (35) and (36) to define γ_p as the canonical map,

$$\begin{array}{c} I^{\otimes(m_0+1)} \otimes k\big[\mathrm{Epi}_{\Delta S_+}([m_0],[m_1])\big] \otimes \ldots \otimes k\big[\mathrm{Epi}_{\Delta S_+}([m_{p-1}],[m_p])\big] \\ \Big\downarrow \\ I^{\otimes(m_0+1)} \otimes_{kG_0} k\big[\mathrm{Epi}_{\Delta S_+}([m_0],[m_1])\big] \otimes_{kG_1} \ldots \otimes_{kG_{p-1}} k\big[\mathrm{Epi}_{\Delta S_+}([m_{p-1}],[m_p])\big] \end{array}$$

We shall prove that γ_* induces an isomorphism in homology by induction on p. First, if $p = 0$, then $\mathscr{B}_*^{(m_0)} = I^{\otimes(m_0+1)}$, and concentrated in degree 0. Moreover, γ_0 is the identity $\mathscr{B}_0^{(m_0)} \to \mathscr{M}_0^{(m_0)}$.

Next assume $\gamma_* : \mathscr{B}_*^{(m_0,\ldots,m_{p-1})} \to \mathscr{M}_*^{(m_0,\ldots,m_{p-1})}$ induces an isomorphism in homology for any string of p numbers $m_0 > m_1 > \ldots > m_{p-1}$. Now assume $m_p < m_{p-1}$. Let $G = \Sigma_{m_{p-1}+1}$. As graded k-module, there is a degree-preserving isomorphism:

$$\theta_* \ : \ \mathscr{B}_*^{(m_0,\ldots,m_{p-1})} \otimes_{kG} E_*G \otimes k\big[G \times \mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\big] \longrightarrow \mathscr{B}_*^{(m_0,\ldots,m_{p-1},m_p)} \tag{37}$$

where the degree of an element $u \otimes (g_0,\ldots,g_n) \otimes (g,\phi)$ is defined recursively: $deg\,(u \otimes (g_0,\ldots,g_n) \otimes (g,\phi)) := deg(u) + n + 1$, and $deg(u_0) = 0$ for any $u_0 \in \mathscr{B}_n^{(m_0)}$. θ_* is defined on generators, $u \otimes (g_0, g_1, \ldots, g_n) \otimes (g,\phi)$, by letting g_0 act on the right of u, then appending the remaining group elements $g_1,\ldots,g_n$ onto the chain as automorphisms, and finally appending the morphism (g,ϕ) to the end. Explicitly,

$$\theta_*\,(u \otimes (g_0,g_1,\ldots,g_n) \otimes (g,\phi)) := (u.g_0) \, * \, [m_{p-1}] \xrightarrow{g_1} [m_{p-1}] \xrightarrow{g_2} \ldots \xrightarrow{g_n} [m_{p-1}] \xrightarrow{(\phi,g)} [m_p],$$

where $v * w$ is the concatenation of chains v and w (The final target of the chain v must agree with the first source of the chain w).

If we define a right action of Σ_{m_p+1} on $k\big[G \times \mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\big]$ via $(g,\phi).h := \big(gh^{\phi}, \phi^h\big)$, then θ_* becomes a map of right $(k\Sigma_{m_p+1})$-modules, since the action defined above simply amounts to post-composition of the morphism (ϕ,g) with (id,h).

Observe that $\mathscr{B}_*^{(m_0,\ldots,m_{p-1})} \otimes_{kG} E_*G \otimes k\big[G \times \mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\big]$ is a chain complex, with differential defined on generators by,

$$\partial\big(u \otimes (g_0,\ldots,g_n) \otimes (g,\phi)\big) \ = \ \partial^B(u) \otimes (g_0,\ldots,g_n) \otimes (g,\phi) + (-1)^{deg(u)} u \otimes \partial^{EG}\big((g_0,\ldots,g_n) \otimes (g,\phi)\big).$$

Note, the n^{th} face map of $E_nG \otimes k\big[G \times \mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\big]$ is defined by: $\partial_n\big((g_0,\ldots,g_n) \otimes (g,\phi)\big) = (g_0,\ldots,g_{n-1}) \otimes (g_n g,\phi)$. It is straightfoward but tedious to verify that θ_* is a chain map with respect to this differential.

θ_* has a two-sided $\Sigma^{\mathrm{op}}_{m_p+1}$-equivariant inverse, defined by sending

$$u * [m_{p-1}] \stackrel{g_1}{\to} [m_{p-1}] \stackrel{g_2}{\to} \ldots \stackrel{g_n}{\to} [m_{p-1}] \stackrel{(\phi,g)}{\to} [m_p] \ \mapsto \ u \otimes (\mathrm{id}, g_1, g_2, \ldots, g_n) \otimes (g,\phi).$$

Thus, θ_* is a chain isomorphism.

The next step in this proof is to prove a chain homotopy equivalence,

$$\begin{array}{c} \mathscr{B}_*^{(m_0,\ldots,m_{p-1})} \otimes_{kG} E_*G \otimes k\big[G \times \mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\big] \\ \simeq\Big\downarrow \\ \mathscr{B}_*^{(m_0,\ldots,m_{p-1})} \otimes k\big[\mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\big] \end{array}$$

To that end, we shall define chain maps F_* and G_* between the two complexes. Let $\mathscr{U}_* := \mathscr{B}_*^{(m_0,\ldots,m_{p-1})}$, and $S := \mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])$, and define maps

$$F_* \ : \ \mathscr{U}_* \otimes_{kG} E_*G \otimes k[G \times S] \longrightarrow \mathscr{U}_* \otimes k[S], \tag{38}$$

$$F_*\big(u \otimes (g_0,\ldots,g_n) \otimes (g,\phi)\big) \ := \ \begin{cases} u.(g_0 g) \otimes \phi, & \text{if } n = 0 \\ 0, & \text{if } n > 0. \end{cases}$$

$$G_* \ : \ \mathscr{U}_* \otimes k[S] \to \mathscr{U}_* \otimes_{kG} E_*G \otimes k[G \times S] \tag{39}$$

is the composite, $\mathscr{U}_* \otimes k[S] \xrightarrow{\cong} \mathscr{U}_* \otimes_{kG} G \otimes k[S] = \mathscr{U}_* \otimes_{kG} E_0 G \otimes k[S] \hookrightarrow \mathscr{U}_* \otimes_{kG} E_* G \otimes k[G \times S]$. The last map is induced by inclusions $E_0 G \hookrightarrow E_* G$ and $k[S] \hookrightarrow k[G \times S]$.

Now, $F_* G_*$ is the identity, and $G_* F_* \simeq \mathrm{id}$ via the homotopy, $h_* : u \otimes (g_0, \ldots, g_n) \otimes (g, \phi) \mapsto (-1)^{deg(u)+n} u \otimes (g_0, \ldots, g_n, g) \otimes (\mathrm{id}, \phi)$. The verification of this claim is rather tedious, but easy.

Since $\mathscr{M}_*^{(m_0,\ldots,m_{p-1})} \otimes k\big[\mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\big] \cong \mathscr{M}_*^{(m_0,\ldots,m_p)}$, the proof of Prop 69 follows from the fact that γ_* decomposes as the following chain of isomorphisms and homotopy equivalences (each of which is also $\Sigma^{\mathrm{op}}_{m_p+1}$-equivariant):

$$\mathscr{B}_*^{(m_0,\ldots,m_p)} \xrightarrow{\theta_*^{-1}} \mathscr{B}_*^{(m_0,\ldots,m_{p-1})} \otimes_{kG} E_* G \otimes k\left[G \times \mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\right] \xrightarrow{F_*}$$

$$\mathscr{B}_*^{(m_0,\ldots,m_{p-1})} \otimes k\left[\mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\right] \xrightarrow{\gamma_* \otimes \mathrm{id}} \mathscr{M}_*^{(m_0,\ldots,m_{p-1})} \otimes k\left[\mathrm{Epi}_{\Delta_+}([m_{p-1}],[m_p])\right] \xrightarrow{\cong} \mathscr{M}_*^{(m_0,\ldots,m_p)}$$

□

Now, we may prove Lemma 67. Let $G = \Sigma_{m_p+1}$. Observe,

$$E^0_{p,q} \cong \bigoplus_{m_0 > \ldots > m_p} \bigoplus_{s+t=q} \mathscr{B}_s^{(m_0,\ldots,m_p)} \otimes_{kG} E_t G,$$

with differential corresponding exactly to the vertical differential defined for E^0. Note, the outer direct sum respects the differential d^0, so the E^1 term given by:

$$E^1_{p,q} = H_{p+q}(E^0_{p,*}) \cong \bigoplus_{m_0 > \ldots > m_p} H_{p+q}\left(\mathscr{B}_*^{(m_0,\ldots,m_p)} \otimes_{kG} E_* G\right), \tag{40}$$

where we view $\mathscr{B}_*^{(m_0,\ldots,m_p)} \otimes_{kG} E_* G$ as a double complex. In what follows, let $(m_0, \ldots, m_p)$ be fixed. In order to take the homology of the double complex, we set up another spectral sequence. From the discussion above, the total differential is given by $\partial^{total} = d^v + d^h$, where $d^v\,(u \otimes (g_0, \ldots, g_t)) := \partial^B(u) \otimes (g_0, \ldots, g_t)$, and $d^h\,(u \otimes (g_0, \ldots, g_t)) := (-1)^{deg(u)} u \otimes \partial^{EG}(g_0, \ldots, g_t)$. Thus, there is a spectral sequence $\{\widetilde{E}^r_{*,*}, d^r\}$ with $\widetilde{E}^2 \cong H_{*,*}\left(H_*\left(\mathscr{B}_*^{(m_0,\ldots,m_p)} \otimes_{kG} E_* G,\, d^h\right),\, d^v\right) \Rightarrow H_*\left(\mathscr{B}_*^{(m_0,\ldots,m_p)} \otimes_{kG} E_* G\right)$.

Let us examine the $\widetilde{E}^2$ term more closely. Let t be fixed, and take the horizontal differential of the original double complex. We obtain $\widetilde{E}^1_{*,t} = H_*\left(\mathscr{B}_*^{(m_0,\ldots,m_p)} \otimes_{kG} E_t G,\, d^h\right) \cong H_*\left(\mathscr{B}_*^{(m_0,\ldots,m_p)}\right) \otimes_{kG} E_t G$, since $E_t G$ is flat as left kG-module (in fact, $E_t G$ is free). Then, by Prop. 69, $\widetilde{E}^1_{*,t} \cong H_*(\mathscr{M}_*^{(m_0,\ldots,m_p)}) \otimes_{kG} E_t G$.

$$= \begin{cases} I^{\otimes(m_0+1)} \otimes k\left[\prod_{i=1}^p \mathrm{Epi}_{\Delta_+}\left([m_{i-1}],[m_i]\right)\right] \otimes_{kG} E_t G, & \text{in degree } p \\ 0, & \text{in degrees different from } p \end{cases}$$

So, the only groups that survive are concentrated in column p. Taking the vertical differential now amounts to computing the G^{op}-equivariant homology of $I^{\otimes(m_0+1)} \otimes k\left[\prod_{i=1}^p \mathrm{Epi}_{\Delta_+}\big([m_{i-1}],[m_i]\big)\right]$, so

$$\widetilde{E}^2_{p,t} \cong H_t\left(G^{\mathrm{op}};\ I^{\otimes(m_0+1)} \otimes k\left[\prod_{i=1}^p \mathrm{Epi}_{\Delta_+}\left([m_{i-1}],[m_i]\right)\right]\right).$$

Since $\widetilde{E}^2_{s,t} = 0$ for $s \neq p$, the sequence collapses here. Thus,

$$H_{p+q}\left(\mathscr{B}_*^{(m_0,\ldots,m_p)} \otimes_{kG} E_* G\right) \cong H_q\left(G^{\mathrm{op}};\ I^{\otimes(m_0+1)} \otimes k\left[\prod_{i=1}^p \mathrm{Epi}_{\Delta_+}\left([m_{i-1}],[m_i]\right)\right]\right)$$

Putting this information back into eq. (40), we obtain the desired isomorphism:

$$E^1_{p,q} \cong \bigoplus_{m_0 > \ldots > m_p} H_q\left(G^{\mathrm{op}}\ ;\ I^{\otimes(m_0+1)} \otimes k\left[\prod_{i=1}^p \mathrm{Epi}_{\Delta_+}\left([m_{i-1}],[m_i]\right)\right]\right).$$

A final piece of information needed in order to use Lemma 67 for computation is a description of the horizontal differential $d^1_{p,q}$ on $E^1_{p,q}$. This map is induced from the differential d on $\mathscr{Y}_*$, and reduces the

filtration degree by 1. Thus, it is the sum of face maps that combine strict epimorphisms. Let

$$[u] \in \bigoplus H_q\left(\Sigma^{\mathrm{op}}_{m_p+1};\ I^{\otimes(m_0+1)} \otimes k\left[\prod_{i=1}^{p} \mathrm{Epi}_{\Delta_+}([m_{i-1}],[m_i])\right]\right)$$

be represented by a chain, $u = Y \otimes (\phi_1, \phi_2, \ldots, \phi_p) \otimes (g_0, \ldots, g_q)$. Then, the face maps d_i of $d^1_{p,q}$ are defined for $0 \leq i < p$ by:

$$d_i(u) := \begin{cases} (\phi_1)_*(Y) \otimes (\phi_2, \ldots, \phi_p) \otimes (g_0, \ldots, g_q), & \text{for } i = 0 \\ Y \otimes (\phi_1, \ldots, \phi_{i+1}\phi_i, \ldots, \phi_p) \otimes (g_0, \ldots, g_q), & \text{for } 0 < i < p. \end{cases}$$

The last face map d_p has the effect of removing the morphism ϕ_p by iteratively commuting it past any group elements to the right of it: $d_p(u) = Y \otimes (\phi_1, \ldots, \phi_{p-1}) \otimes (g'_0, \ldots, g'_q)$, where $g'_i = g_i^{\phi_p^{g_0 g_1 \cdots g_{i-1}}}$. Note that d_p involves a change of group from Σ_{m_p} to $\Sigma_{m_{p-1}}$.

Proposition 70. *The spectral sequence $E^r_{p,q}$ above collapses at $r = 2$.*

Proof. This proof relies on the fact that the differential d on $\mathscr{Y}_*$ cannot reduce the filtration degree by more than 1. Explicitly, we shall show that $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial for $r \geq 2$.

d^r is induced by d in the following way. Let $Z^r_p = \{x \in \mathscr{F}_p\mathscr{Y}_* \,|\, d(x) \in \mathscr{F}_{p-r}\mathscr{Y}_*\}$. Then $E^r_{p,*} = Z^r_p/(Z^{r-1}_{p-1} + dZ^{r-1}_{p+r-1})$. Now, d maps $Z^r_p \to Z^r_{p-r}$ and $Z^{r-1}_{p-1} + dZ^{r-1}_{p+r-1} \to dZ^{r-1}_{p-1}$. Hence, there is an induced map $\overline{d}$ making the square below commute. d^r is obtained as the composition of $\overline{d}$ with a projection onto $E^r_{p-r,*}$.

$$\begin{tikzcd} Z^r_p \arrow[r, "d"] \arrow[d, two heads, "\pi_1"'] & Z^r_{p-r} \arrow[d, two heads, "\pi_2"] \\ Z^r_p/(Z^{r-1}_{p-1} + dZ^{r-1}_{p+r-1}) \arrow[r, "\overline{d}"] \arrow[d, equal] & Z^r_{p-r}/dZ^{r-1}_{p-1} \arrow[d, two heads, "\pi'"] \\ E^r_{p,*} \arrow[dr, "d^r"] & Z^r_{p-r}/(Z^{r-1}_{p-r-1} + dZ^{r-1}_{p-1}) \arrow[d, equal] \\ & E^r_{p-r,*} \end{tikzcd}$$

In our case, $x \in Z^r_p$ is a sum of the form $x = \sum_{q\geq 0} a_i\,(Y \otimes (f_1, f_2, \ldots, f_q))$, where $a_i \neq 0$ for only finitely many i, and the sum extends over all symbols $Y \otimes (f_1, f_2, \ldots, f_q)$ with $Y \in B^{sym_+}_* I$, $f_j \in \mathrm{Epi}\Delta\mathrm{S}_+$ composable maps, and at most p of the f_j maps are strict epimorphisms. The image of x under π_1 looks like $\pi_1(x) = \sum_{q\geq 0} a_i\,[Y \otimes (f_1, f_2, \ldots, f_q)]$, where exactly p of the f_j maps are strictly epic. There are, of course, other relations present as well − those arising from modding out by dZ^{r-1}_{p+r-1}. Consider, $\overline{d}\pi_1(x)$. This should be the result of lifting $\pi_1(x)$ to a representative in Z^r_p, then applying $\pi_2 d$. One such representative is $y = \sum_{q\geq 0} a_i\,(Y \otimes (f_1, f_2, \ldots, f_q))$, in which each symbol $Y \otimes (f_1, f_2, \ldots, f_q)$ has exactly p strict epimorphisms. Now, $d(y)$ is the sum $\sum_{q\geq 0} b_i\,(Z \otimes (g_1, g_2, \ldots, g_{q-1}))$, where each symbol $Z \otimes (g_1, g_2, \ldots, g_{q-1})$ has either p or $p-1$ strict epimorphisms. Thus, if $r \geq 2$, then $d(y) \in Z^r_{p-r}$ implies $d(y) = 0$. But then, $\overline{d}\pi_1(x) = \pi_2 d(y) = 0$, and $d^r = \pi'\overline{d}$ is trivial. □

7.3. **Implications in Characteristic 0.** If k is a field of characteristic 0, then for any finite group G and kG-module M, $H_q(G, M) = 0$ for all $q > 0$ (see [1], for example). Thus, by Lemma 67, the E^1 term of the spectral sequence would be concentrated in row 0, and

$$E^1_{p,0} \cong \bigoplus_{m_0 > \ldots > m_p} \left(I^{\otimes(m_0+1)} \otimes k\left[\prod_{i=1}^{p} \mathrm{Epi}_{\Delta_+}([m_{i-1}],[m_i])\right]\right)/\Sigma^{\mathrm{op}}_{m_p+1},$$

that is, the group of co-invariants of the coefficient group, under the right-action of Σ_{m_p+1}.

Since E^1 is concentrated on a row, the spectral sequence collapses at this term. Hence for the k-algebra A, with augmentation ideal I,

$$HS_*(A) = H_*\left(\bigoplus_{p\geq 0}\bigoplus_{m_0>\ldots>m_p}\left(I^{\otimes(m_0+1)}\otimes k\left[\prod_{i=1}^{p}\mathrm{Epi}_{\Delta_+}([m_{i-1}],[m_i])\right]\right)/\Sigma^{\mathrm{op}}_{m_p+1},\ d^1\right). \tag{41}$$

This complex is still rather unwieldy as the E^1 term is infinitely generated in each degree. In the next chapter, we shall see another spectral sequence that is more computationally useful.

8. A Second Spectral Sequence

8.1. **Reduced Symmetric Homology.** Again, we shall assume A is a k-algebra equipped with an augmentation, and whose augmentation ideal is I. Assume further that I is free as k-module, with countable basis X. Denote by $B_*^{sym}I$, the restriction of $B_*^{sym_+}I$ to ΔS. That is, $B_n^{sym}I := I^{\otimes(n+1)}$ for all $n\geq 0$. Let $\widetilde{\mathscr{Y}_*} := k[N(-\setminus \mathrm{Epi}\Delta S)]\otimes_{\mathrm{Epi}\Delta S} B_*^{sym}I$. Observe that there is a splitting $\mathscr{Y}_*^{epi}A \cong \widetilde{\mathscr{Y}_*}\oplus k[N(*)]$, where $\mathscr{Y}_*^{epi}A$ is the complex (31) of Section 6, and $*$ is the trivial subcategory of $\mathrm{Epi}\Delta S_+$ consisting of the object $[-1]$ and morphism $\mathrm{id}_{[-1]}$. The fact that I is an ideal ensures that this splitting passes to homology. Hence, we have $HS_*(A)\cong H_*(\widetilde{\mathscr{Y}_*})\oplus k_0$, where k_0 is the graded k-module consisting of k concentrated in degree 0.

Definition 71. The *reduced symmetric homology* of A is defined, $\widetilde{HS}_*(A) := H_*(\widetilde{\mathscr{Y}_*})$.

Now, since $I = k[X]$ as k-module and $B_n^{sym}I = k[X]^{\otimes(n+1)}$, $\widetilde{\mathscr{Y}_*}$ is generated, as a k-module, by elements of the form $[m_0]\twoheadrightarrow[m_1]\twoheadrightarrow\ldots\twoheadrightarrow[m_p]\otimes(x_0\otimes x_1\otimes\ldots\otimes x_{m_0})$, with $x_i\in X$. Analogous to Eq. (35), denote

$$\widetilde{\mathscr{B}}^{(m_0,m_1,\ldots,m_q)} := k[X]^{\otimes(m_0+1)}\otimes\bigotimes_{i=1}^{q}k\left[\mathrm{Epi}_{\Delta S}([m_{i-1}],[m_i])\right]$$

Then we may observe, $\widetilde{\mathscr{Y}_q}\cong\bigoplus_{m_0\geq\ldots\geq m_q}\widetilde{\mathscr{B}}^{(m_0,m_1,\ldots,m_q)}$. The face maps are given on generators as follows:

$$d_i(Y\otimes f_1\otimes f_2\otimes\ldots\otimes f_q) = \begin{cases} f_1(Y)\otimes f_2\otimes\ldots\otimes f_q, & \text{for } i=0\\ Y\otimes f_1\otimes\ldots\otimes(f_{i+1}f_i)\otimes\ldots\otimes f_q, & \text{for } 1\leq i<q\\ Y\otimes f_1\otimes\ldots\otimes f_{q-1} & \text{for } i=q.\end{cases}$$

8.2. **Filtering by Degree.** Consider a filtration $\mathscr{G}_*$ of $\widetilde{\mathscr{Y}_*}$ given by,

$$\mathscr{G}_p\widetilde{\mathscr{Y}_q} = \bigoplus_{p\geq m_0\geq\ldots\geq m_q}\widetilde{\mathscr{B}}^{(m_0,m_1,\ldots,m_q)}.$$

Observe that $\mathscr{G}_*$ filters the complex by degree of $Y\in B_*^{sym}I$. The only face map that can potentially change the degree of Y is d_0, and since all morphisms are epic, d_0 can only reduce the degree. Thus, $\mathscr{G}_*$ is compatible with the differential of $\widetilde{\mathscr{Y}_*}$. The filtration quotients are easily described:

$$E^0_{p,q} = \bigoplus_{p=m_0\geq\ldots\geq m_q}\widetilde{\mathscr{B}}^{(m_0,m_1,\ldots,m_q)}.$$

E^0 splits into a direct sum based on the product of x_i's in $(x_0,\ldots,x_p)\in X^{p+1}$. For $u\in X^{p+1}$, let P_u be the set of all distinct permutations of u. Then,

$$E^0_{p,q} = \bigoplus_{u\in X^{p+1}/\Sigma_{p+1}}\left(\bigoplus_{p=m_0\geq\ldots\geq m_q}\bigoplus_{w\in P_u}w\otimes k\left[\prod_{i=1}^{q}\mathrm{Epi}_{\Delta S}([m_{i-1}],[m_i])\right]\right).$$

8.3. **The Categories $\widetilde{\mathbb{S}}_p$, $\widetilde{\mathbb{S}}'_p$, $\mathbb{S}_p$ and $\mathbb{S}'_p$.** Before proceeding with the main theorem of this section, we must define four related categories. In the definitions that follow, let $\{z_0,z_1,z_2,\ldots z_p\}$ be a set of formal non-commuting indeterminates.

Definition 72. $\widetilde{\mathbb{S}}_p$ is the category with objects formal tensor products $Z_0\otimes\ldots\otimes Z_s$, where each Z_i is a non-empty product of z_i's, and every one of $z_0,z_1,\ldots,z_p$ occurs exactly once in the tensor product. There is a unique morphism $Z_0\otimes\ldots\otimes Z_s\to Z'_0\otimes\ldots\otimes Z'_t$, if and only if the tensor factors of the latter are products of the factors of the former in some order. In such a case, there is a unique $\beta\in\mathrm{Epi}\Delta S$ so that $\beta_*(Z_0\otimes\ldots\otimes Z_s) = Z'_0\otimes\ldots\otimes Z'_t$.

$\widetilde{\mathcal{S}}_p$ has initial objects $\sigma_*(z_0 \otimes z_1 \otimes \ldots \otimes z_p)$, for $\sigma \in \Sigma_{p+1}^{\mathrm{op}}$, so $N\widetilde{\mathcal{S}}_p$ is a contractible complex. Let $\widetilde{\mathcal{S}}'_p$ be the full subcategory of $\widetilde{\mathcal{S}}_p$ with all objects $\sigma_*(z_0 \otimes \ldots \otimes z_p)$ deleted.

Let $\mathcal{S}_p$ be a skeletal category equivalent to $\widetilde{\mathcal{S}}_p$. In fact, we may make $\mathcal{S}_p$ the quotient category, identifying each object $Z_0 \otimes \ldots \otimes Z_s$ with any permutation of its tensor factors, and identifying morphisms ϕ and ψ if their source and target are equivalent. This category has nerve $N\mathcal{S}_p$ homotopy-equivalent to $N\widetilde{\mathcal{S}}_p$. Now, $\mathcal{S}_p$ is a poset with unique initial object, $z_0 \otimes \ldots \otimes z_p$. Let $\mathcal{S}'_p$ be the full subcategory (subposet) of $\mathcal{S}_p$ obtained by deleting the object $z_0 \otimes \ldots \otimes z_p$. Clearly, $\mathcal{S}'_p$ is a skeletal category equivalent to $\widetilde{\mathcal{S}}'_p$.

8.4. **Main Theorem.**

Theorem 73. *There is spectral sequence converging (weakly) to $\widetilde{H}S_*(A)$ with*

$$E^1_{p,q} \cong \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q}\left(EG_u \ltimes_{G_u} |N\mathcal{S}_p/N\mathcal{S}'_p|; k\right),$$

where G_u is the isotropy subgroup for the chosen representative of $u \in X^{p+1}/\Sigma_{p+1}$.

Recall, for a group G, right G-space X, and left G-space Y, $X \ltimes_G Y$ denotes the *equivariant half-smash product.* If $*$ is a chosen basepoint for Y having trivial G-action, then $X \ltimes_G Y := (X \times_G Y)/(X \times_G *) = X \times Y/\approx$, with equivalence relation defined by $(x.g, y) \approx (x, g.y)$ and $(x, *) \approx (x', *)$ for all $x, x' \in X$, $y \in Y$ and $g \in G$ (cf. [15]). In our case, X is of the form EG, with canonical underlying complex E_*G, equipped with a right G-action, $(g_0, g_1, \ldots, g_n).g = (g_0, g_1, \ldots, g_n g)$.

Observe, both $N\widetilde{\mathcal{S}}_p$ and $N\widetilde{\mathcal{S}}'_p$ carry a left Σ_{p+1}-action (hence also a G_u-action). The action is defined on 0-chains $Z_0 \otimes \ldots \otimes Z_s$ by permutation of the individual indeterminates, $z_0, z_1, \ldots, z_p$. This action extends to n-chains in the straightforward manner.

Define for each $u \in X^{p+1}/\Sigma_{p+1}$, the following subcomplex of $E^0_{p,q}$:

$$\mathscr{M}_u := \bigoplus_{p = m_0 \geq \ldots \geq m_q} \bigoplus_{w \in P_u} w \otimes k\left[\prod_{i=1}^{q} \mathrm{Epi}_{\Delta S}\left([m_{i-1}], [m_i]\right)\right]$$

Lemma 74. *There is a chain-isomorphism, $\left(N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p\right)/G_u \stackrel{\cong}{\to} \mathscr{M}_u$.*

Proof. Let C_j denote objects of $\widetilde{\mathcal{S}}_p$. As above, we may view each C_j as a morphism of ΔS. By abuse of notation, let C_j also represent a 0-cell of $N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p$. Denote the chosen representative of u again by u (We view $u = (x_{i_0}, x_{i_1}, \ldots, x_{i_p}) \in X^{p+1}/\Sigma_{p+1}$ as represented by a $(p+1)$-tuple whose indices are in non-decreasing order).

First define a map $N\widetilde{\mathcal{S}}_p \longrightarrow \mathscr{M}_u$ on n-cells by:

$$\alpha_* : \left(C_0 \stackrel{\phi_1}{\to} \ldots \stackrel{\phi_q}{\to} C_q\right) \mapsto \left(C_0(u) \otimes \phi_1 \otimes \ldots \otimes \phi_q\right). \tag{42}$$

(Note, the notation $C_0(u)$ is used in place of the more correct $(C_0)_*(u)$ in order to avoid clutter of notation.) I claim α_* factors through $N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p$. Indeed, if $\left(C_0 \stackrel{\phi_1}{\to} \ldots \stackrel{\phi_q}{\to} C_q\right) \in N\widetilde{\mathcal{S}}'_p$, then we cannot have $C_0 = \sigma(z_0 \otimes \ldots \otimes z_p)$ for any symmetric group element σ. That is, C_0, viewed as a morphism, must be strictly epic. Then the degree of of $C_0(u)$ is strictly less than the degree of u, which would make $(C_0(u) \otimes \phi_1 \otimes \ldots \otimes \phi_q)$ trivial in $E^0_{p,q}$.

The map α_* then factors through $\left(N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p\right)/G_u$, since if $\gamma \in G_u$, then γ corresponds to an automorphism $g \in \Sigma_{p+1}^{\mathrm{op}}$, and by definition, we have: $\gamma.\left(C_0 \stackrel{\phi_1}{\to} \ldots \stackrel{\phi_q}{\to} C_q\right) = \left(C_0 \circ g \stackrel{\phi_1}{\to} \ldots \stackrel{\phi_q}{\to} C_q \circ g\right) \mapsto (C_0(g(u)) \otimes \phi_1 \otimes \ldots \otimes \phi_q) = (C_0(u) \otimes \phi_1 \otimes \ldots \otimes \phi_q)$. (Note, $g(u) = u$ follows from the fact that $\gamma \in G_u$, the isotropy subgroup for u).

For the opposite direction, we begin with a generator of the form, $w \otimes \phi_1 \otimes \ldots \otimes \phi_q$ for some $w \in P_u$. Let $\tau \in \Sigma_{p+1}$ so that $w = t(u)$, where $t \in \Sigma_{p+1}^{\mathrm{op}}$ corresponds to τ. Define a map sending,

$$\beta_* : (w \otimes \phi_1 \otimes \ldots \otimes \phi_q) \mapsto \left(t \stackrel{\phi}{\to} \phi_1 t \stackrel{\phi_2}{\to} \ldots \stackrel{\phi_q}{\to} \phi_q \cdots \phi_1 t\right). \tag{43}$$

We must check that the definition of β_* does not depend on choice of τ. Indeed, if $w = s(u)$ also, then $u = s^{-1}t(u)$, hence $s^{-1}t \in G_u^{\mathrm{op}}$. Thus,

$$\left(s \xrightarrow{\phi} \phi_1 s \xrightarrow{\phi_2} \dots \xrightarrow{\phi_q} \phi_q \dots \phi_1 s\right) \;\approx\; s^{-1}t.\left(s \xrightarrow{\phi} \phi_1 s \xrightarrow{\phi_2} \dots \xrightarrow{\phi_q} \phi_q \cdots \phi_1 s\right) = \left(t \xrightarrow{\phi} \phi_1 t \xrightarrow{\phi_2} \dots \xrightarrow{\phi_q} \phi_q \cdots \phi_1 t\right).$$

The maps α_* and β_* are clearly inverse to one another. All that remains is to verify that they are chain maps. We need only check compatibility with the zeroth face maps in either case, since the i^{th} face maps (for $i > 0$) simply compose the morphisms ϕ_{i+1} and ϕ_i in either chain complex. Consider α_* first. (The zeroth face maps of either complex will be denoted d_0).

First consider the map $alpha_*$.

$$\begin{tikzcd}
\left(C_0 \xrightarrow{\phi_1} C_1 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_q} C_q\right) \arrow[rr, mapsto, "d_0"] \arrow[d, mapsto, "\alpha_*"'] & & \left(C_1 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_q} C_q\right) \arrow[d, mapsto, "\alpha_*"'] \\
(C_0(u) \otimes \phi_1 \otimes \phi_2 \otimes \dots \otimes \phi_q) \arrow[r, mapsto, "d_0"] & (\phi_1 C_0(u) \otimes \phi_2 \otimes \dots \otimes \phi_q) \arrow[r, equal] & (C_1(u) \otimes \phi_2 \otimes \dots \otimes \phi_q)
\end{tikzcd}$$

The equality in the lower right of the diagram is simply a restatement that $C_0 \xrightarrow{\phi_1} C_1$ is a morphism of $\widetilde{\mathcal{S}}_p$.

For the reverse direction, assume $w = t(u)$ as above

$$\begin{tikzcd}
(w \otimes \phi_1 \otimes \dots \otimes \phi_q) \arrow[r, mapsto, "d_0"] \arrow[d, mapsto, "\beta_*"'] & (\phi_1(w) \otimes \phi_2 \dots \otimes \phi_q) \arrow[d, mapsto, "\beta_*"'] \\
\left(t \xrightarrow{\phi} \phi_1 t \xrightarrow{\phi_2} \dots \xrightarrow{\phi_q} \phi_q \cdots \phi_1 t\right) \arrow[r, mapsto, "d_0"] & \left(\phi_1 t \xrightarrow{\phi_2} \dots \xrightarrow{\phi_q} \phi_q \cdots \phi_1 t\right)
\end{tikzcd}$$

□

Using this lemma, we identify $\mathscr{M}_u$ with the orbit complex $\left(N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p\right)/G_u$. Now, the complex $N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p$ is a free G_u-complex, so we have an isomorphism: $H_*\left(\left(N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p\right)/G_u\right) \cong H_*^{G_u}\left(N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p\right)$. (*i.e.*, G_u-equivariant homology. See [1] for details). Then, by definition, $H_*^{G_u}\left(N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p\right) = H_*\left(G_u, N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p\right)$, which may be computed using the free resolution, E_*G_u of k as right G_u-module. The resulting complex $k[E_*G_u] \otimes_{kG_u} k\left[N\widetilde{\mathcal{S}}_p\right]/k\left[N\widetilde{\mathcal{S}}'_p\right]$ is a double complex isomorphic to the quotient of two double complexes, namely:

$$\left(k[E_*G_u] \otimes_{kG_u} k\left[N\widetilde{\mathcal{S}}_p\right]\right) / \left(k[E_*G_u] \otimes_{kG_u} k\left[N\widetilde{\mathcal{S}}'_p\right]\right) \;\cong\; k\left[\left(E_*G_u \times_{G_u} N\widetilde{\mathcal{S}}_p\right) / \left(E_*G_u \times_{G_u} N\widetilde{\mathcal{S}}'_p\right)\right].$$

This last complex may be identified with the simplicial complex of the space,

$$\left(EG_u \times_{G_u} |N\widetilde{\mathcal{S}}_p|\right) / \left(EG_u \times_{G_u} |N\widetilde{\mathcal{S}}'_p|\right) \cong EG_u \ltimes_{G_u} |N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p|.$$

The last piece of the puzzle involves simplifying the spaces $|N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p|$. Since $\mathcal{S}$ is a skeletal subcategory of $\widetilde{\mathcal{S}}$, there is an equivalence of categories $\widetilde{\mathcal{S}} \simeq \mathcal{S}$, inducing a homotopy equivalence of complexes (hence also of spaces) $|N\widetilde{\mathcal{S}}| \simeq |N\mathcal{S}|$. Note that $N\mathcal{S}$ inherits a G_u-action from $N\widetilde{\mathcal{S}}$, and the map $\widetilde{\mathcal{S}} \to \mathcal{S}$ is G_u-equivariant.

Proposition 75. *There are weak equivalences,* $EG_u \times_{G_u} |N\widetilde{\mathcal{S}}_p| \to EG_u \times_{G_u} |N\mathcal{S}_p|$ *and* $EG_u \times_{G_u} |N\widetilde{\mathcal{S}}'_p| \to EG_u \times_{G_u} |N\mathcal{S}'_p|$*, inducing a weak equivalence* $EG_u \ltimes_{G_u} |N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p| \to EG_u \ltimes_{G_u} |N\mathcal{S}_p/N\mathcal{S}'_p|$.

Proof. The case $p > 2$ will be handled first. Consider the fibration $X \to EG \times_G X \to BG$ associated to a group G and path-connected G-space X. The resulting homotopy sequence breaks up into isomorphisms $0 \to \pi_i(X) \xrightarrow{\cong} \pi_i(EG \times_G X) \to 0$ for $i \geq 2$ and a short exact sequence $0 \to \pi_1(X) \to \pi_1(EG \times_G X) \to G \to 0$. If there is a G-equivariant homotopy-equivalence $f : X \to Y$ for a path-connected G-space Y, then for $i \geq 2$,

we have isomorphisms $\pi_i(EG \times_G X) \leftarrow \pi_i(X) \xrightarrow{f_*} \pi_i(Y) \rightarrow \pi_i(EG \times_G Y)$, and a diagram corresponding to $i = 1$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \pi_1(X) \arrow[r] \arrow[d, "f_*"'] & \pi_1(EG \times_G X) \arrow[r] \arrow[d, "(\mathrm{id}\times f)_*"] & G \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & \pi_1(Y) \arrow[r] & \pi_1(EG \times_G Y) \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

Thus, there is a weak equivalence $EG \times_G X \rightarrow EG \times_G Y$.

So in our case the desired result will be proved if the spaces $|N\widetilde{\mathcal{S}}'_p|$ and $|N\mathcal{S}'_p|$ are path-connected. (Note, $|N\widetilde{\mathcal{S}}_p|$ and $|N\mathcal{S}_p|$ are path-connected because they are contractible). In fact, we need only check $|N\mathcal{S}'_p|$, since this space is homotopy-equivalent to $|N\widetilde{\mathcal{S}}'_p|$.

let $W_0 := z_0 z_1 \otimes z_2 \otimes \ldots \otimes z_p$. This represents a vertex of $N\mathcal{S}'_p$. Suppose $W = Z_0 \otimes \ldots \otimes Z'_i z_0 z_1 Z''_i \otimes \ldots \otimes Z_s$. Then there is a morphism $W_0 \rightarrow W$, hence an edge between W_0 and W.

Next, suppose $W = Z_0 \otimes \ldots \otimes Z'_i z_0 Z''_i z_1 Z'''_i \otimes \ldots \otimes Z_s$. There is a path:

$$\begin{tikzcd}
Z_0 \otimes \ldots \otimes Z'_i z_0 Z''_i z_1 Z'''_i \otimes \ldots \otimes Z_s \arrow[d] \\
Z_0 Z_1 \ldots Z'_i z_0 Z''_i z_1 Z'''_i \ldots Z_s \\
Z_0 Z_1 \ldots Z'_i \otimes z_0 \otimes Z''_i z_1 Z'''_i \ldots Z_s \arrow[u] \arrow[d] \\
z_0 \otimes Z_0 Z_1 \ldots Z'_i Z''_i z_1 Z'''_i \ldots Z_s \\
z_0 \otimes Z_0 Z_1 \ldots Z'_i Z''_i \otimes z_1 Z'''_i \ldots Z_s \arrow[u] \arrow[d] \\
z_0 z_1 Z'''_i \ldots Z_s \otimes Z_0 Z_1 \ldots Z'_i Z''_i \\
W_0 \arrow[u]
\end{tikzcd}$$

Similarly, if $W = Z_0 \otimes \ldots \otimes Z'_i z_1 Z''_i z_0 Z'''_i \otimes \ldots \otimes Z_s$, there is a path to W_0. Finally, if $W = Z_0 \otimes \ldots \otimes Z_s$ with z_0 occurring in Z_i and z_1 occurring in Z_j for $i \neq j$, there is an edge to some W' in which $Z_i Z_j$ occurs, and thus a path to W_0.

The cases $p = 0, 1$ and 2 are handled individually:

Observe that $|N\widetilde{\mathcal{S}}'_0|$ and $|N\mathcal{S}'_0|$ are empty spaces, since $\widetilde{\mathcal{S}}'_0$ has no objects. Hence, $EG_u \times_{G_u} |N\widetilde{\mathcal{S}}'_0| = EG_u \times_{G_u} |N\mathcal{S}'_0| = \emptyset$. Furthermore, any group G_u must be trivial. Thus there is a chain of homotopy equivalences, $EG_u \ltimes_{G_u} |N\widetilde{\mathcal{S}}_0/N\widetilde{\mathcal{S}}'_0| \simeq |N\widetilde{\mathcal{S}}_0| \simeq |N\mathcal{S}_0| \simeq EG_u \ltimes_{G_u} |N\mathcal{S}_p/N\mathcal{S}'_p|$.

Next, since $|N\widetilde{\mathcal{S}}'_1|$ is homeomorphic to $|N\mathcal{S}'_1|$, each space consisting of the two discrete points z_0z_1 and z_1z_0 with the same group action, the proposition is true for $p = 1$ as well.

For $p = 2$, observe that $|N\widetilde{\mathcal{S}}'_2|$ has two connected components, $\widetilde{U}_1$ and $\widetilde{U}_2$ that are interchanged by any odd permutation $\sigma \in \Sigma_3$. Similarly, $|N\mathcal{S}'_2|$ consists of two connected components, U_1 and U_2, interchanged by any odd permutation of Σ_3. Now, resticted to the alternating group, A_3, we certianly have weak equivalences for any subgroup $H_u \subseteq A_3$, $EH_u \times_{H_u} \widetilde{U}_1 \xrightarrow{\simeq} EH_u \times_{H_u} U_1$ and $EH_u \times_{H_u} \widetilde{U}_2 \xrightarrow{\simeq} EH_u \times_{H_u} U_2$. The

action of an odd permutation induces equivariant homeomorphisms $\widetilde{U}_1 \xrightarrow{\cong} \widetilde{U}_2$ and $U_1 \xrightarrow{\cong} U_2$, and so if we have a subgroup $G_u \subseteq \Sigma_3$ generated by $H_u \subseteq A_3$ and a transposition, then the two connected components are identified in an $A\Sigma_3$-equivariant manner. Thus, if G_u contains a transposition, $EG_u \times_{G_u} |N\widetilde{\mathcal{S}}'_2| \cong EH_u \times_{H_u} \widetilde{U}_1 \simeq EH_u \times_{H_u} U_1 \cong EG_u \times_{G_u} |N\mathcal{S}'_2|$. This completes the case $p = 2$ and the proof of Prop. 75. $\square$

Prop. 75 coupled with Lemma 74 produces the required isomorphism in homology, hence proving Thm. 73:

$$E^1_{p,q} = \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q}(\mathscr{M}_u) \cong \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q}\left(EG_u \ltimes_{G_u} |N\widetilde{\mathcal{S}}_p/N\widetilde{\mathcal{S}}'_p|; k\right)$$

$$\cong \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q}\left(EG_u \ltimes_{G_u} |N\mathcal{S}_p/N\mathcal{S}'_p|; k\right).$$

Corollary 76. *If the augmentation ideal of A satisfies $I^2 = 0$, then*

$$HS_n(A) \cong \bigoplus_{p \geq 0} \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_n(EG_u \ltimes_{G_u} N\mathcal{S}_p/N\mathcal{S}'_p; k).$$

Proof. This follows from consideration of the original E^0 term of the spectral sequence. E^0 is generated by chains $Y \otimes \phi_0 \otimes \ldots \otimes \phi_n$, with induced differential d^0, agreeing with the differential d of $\widetilde{\mathscr{Y}_*}$ when ϕ_0 is an isomorphism. When ϕ_0 is a strict epimorphism, however, the zeroth face map of d^0 maps the generator to: $(\phi_0)_*(Y) \otimes \phi_1 \otimes \ldots \otimes \phi_n = 0$, since $(\phi_0)_*(Y)$ would have at least one tensor factor that is the product of two or more elements of I. Thus, d^0 also agrees with d in the case that ϕ_0 is strictly epic. Hence, the spectral sequence collapses at level 1. $\square$

8.5. **The complex $Sym^{(p)}_*$.** Note, for $p > 0$, there are homotopy equivalences $|N\mathcal{S}_p/N\mathcal{S}'_p| \simeq |N\mathcal{S}_p| \vee S|N\mathcal{S}'_p| \simeq S|N\mathcal{S}'_p|$, since $|N\mathcal{S}_p|$ is contractible. $|N\mathcal{S}_p|$ is a disjoint union of $(p+1)!$ p-cubes, identified along certain faces. Geometric analysis of $S|N\mathcal{S}'_p|$, however, seems quite difficult. Fortunately, there is an even smaller chain complex homotopic to $N\mathcal{S}_p/N\mathcal{S}'_p$.

Definition 77. Let $p \geq 0$ and impose an equivalence relation on $k\left[\mathrm{Epi}_{\Delta S}([p],[q])\right]$ generated by:

$$Z_0 \otimes \ldots \otimes Z_i \otimes Z_{i+1} \otimes \ldots \otimes Z_q \approx (-1)^{ab} Z_0 \otimes \ldots \otimes Z_{i+1} \otimes Z_i \otimes \ldots \otimes Z_q,$$

where $Z_0 \otimes \ldots \otimes Z_q$ is a morphism expressed in tensor notation, and $a = deg(Z_i) := |Z_i| - 1$, $b = deg(Z_{i+1}) := |Z_{i+1}| - 1$. Here, $deg(Z)$ is one less than the number of factors of the monomial Z. Indeed, if $Z = z_{i_0} z_{i_1} \ldots z_{i_s}$, then $deg(Z) = s$.

The complex $Sym^{(p)}_*$ is then defined by $Sym^{(p)}_i := k\left[\mathrm{Epi}_{\Delta S}([p],[p-i])\right] / \approx$. The face maps will be defined recursively. On monomials,

$$d_i(z_{j_0} \ldots z_{j_s}) = \begin{cases} 0, & i < 0, \\ z_{j_0} \ldots z_{j_i} \otimes z_{j_{i+1}} \ldots z_{j_s}, \quad , & 0 \leq i < s, \\ 0, & i \geq s. \end{cases}$$

Then, extend d_i to tensor products via:

$$d_i(W \otimes V) = d_i(W) \otimes V + W \otimes d_{i-deg(W)}(V), \tag{44}$$

where W and V are formal tensors in $k\left[\mathrm{Epi}_{\Delta S}([p],[q])\right]$, and $deg(W) = deg(W_0 \otimes \ldots \otimes W_t) := \sum_{k=0}^{t} deg(W_k)$. The boundary map $Sym^{(p)}_n \to Sym^{(p)}_{n-1}$ is then $d = \sum_{i=0}^{n} (-1)^i d_i = \sum_{i=0}^{n-1} (-1)^i d_i$.

Remark 78. The result of applying d_i on any formal tensor product will result in only a single formal tensor product, since in Eq. (44), at most one of the two terms will be non-zero.

Remark 79. There is an action $\Sigma_{p+1} \times Sym^{(p)}_i \to Sym^{(p)}_i$, given by permuting the formal indeterminates z_i. Furthermore, this action is compatible with the differential.

Lemma 80. *$Sym^{(p)}_*$ is chain-homotopy equivalent to $k[N\mathcal{S}_p]/k[N\mathcal{S}'_p]$.*

Proof. Let v_0 represent the common initial vertex of the p-cubes making up $N\mathcal{S}_p$. Then, as cell-complex, $N\mathcal{S}_p$ consists of v_0 together with all corners of the various p-cubes and i-cells for each i-face of the cubes. Thus, $N\mathcal{S}_p$ consists of $(p+1)!$ p-cells with attaching maps $\partial I^p \to (N\mathcal{S}_p)^{p-1}$ defined according to the face maps for $N\mathcal{S}_p$ given above. Note that a chain of $N\mathcal{S}_p$ is non-trivial in $N\mathcal{S}_p/N\mathcal{S}'_p$ if and only if the initial vertex v_0 is included. Thus, any (cubical) k-cell is uniquely determined by the label of the vertex opposite v_0.

Label each top-dimensional cell with the permutation induced on the set $\{0, 1, \ldots, p\}$ by the order of indeterminates in final vertex, $z_{i_0}z_{i_1}\ldots z_{i_p}$. On a given p-cell, for each vertex $Z_0 \otimes \ldots \otimes Z_s$, there is an ordering of the tensor factors so that $Z_0\otimes\ldots\otimes Z_s \to z_{i_0}z_{i_1}\ldots z_{i_p}$ preserves the order of formal indeterminates z_i. Rewrite each vertex of this p-cell in this order. Now, any p-chain $(z_{i_0}\otimes z_{i_1}\otimes\ldots\otimes z_{i_p}) \to \ldots \to z_{i_0}z_{i_1}\ldots z_{i_p}$ is obtained by choosing the order in which to combine the factors. In fact, the p-chains for this cube are in bijection with the elements of the symmetric group S_p, as in the standard decomposition of a p-cube into $p!$ simplices. A given permutation $\{1, 2, \ldots, p\} \mapsto \{j_1, j_2, \ldots, j_p\}$ will represent the chain obtained by first combining $z_{j_0} \otimes z_{j_1}$ into $z_{j_0}z_{j_1}$, then combining $z_{j_1} \otimes z_{j_2}$ into $z_{j_1}z_{j_2}$. In effect, we "erase" the tensor product symbol between $z_{j_{r-1}}$ and z_{j_r} for each j_r in order given by the list above.

We shall declare that the *natural* order of combining the factors will be the one that always combines the last two: $(z_{i_0} \otimes \ldots \otimes z_{i_{p-1}} \otimes z_{i_p}) \to (z_{i_0} \otimes \ldots \otimes z_{i_{p-1}}z_{i_p}) \to (z_{i_0} \otimes \ldots \otimes z_{i_{p-2}}z_{i_{p-1}}z_{i_p}) \to \ldots \to (z_{i_0}\ldots z_{i_p})$. This corresponds to a permutation $\rho := \{1, \ldots, p\} \mapsto \{p, p-1, \ldots, 2, 1\}$, and this chain will be regarded as *positive*. A chain C_σ, corresponding to another permutation, σ, will be regarded as positive or negative depending on the sign of the permutation $\sigma\rho^{-1}$. Finally, the entire p-cell should be identified with the sum $\sum_{\sigma\in S_p} sign(\sigma\rho^{-1})C_\sigma$. It is this sign convention that permits the inner faces of the cube to cancel appropriately in the boundary maps. Thus we have a map on the top-dimensional chains:

$$\theta_p : Sym_p^{(p)} \to \left(k[N\mathcal{S}_p]/k[N\mathcal{S}'_p]\right)_p. \tag{45}$$

Extend the defintion θ_* to arbitrary k-cells by sending the k-chain $Z_0 \otimes \ldots \otimes Z_{p-k}$ to the sum of k-length chains with source $z_0 \otimes \ldots \otimes z_p$ and target $Z_0 \otimes \ldots \otimes Z_{p-k}$ with signs determined by the natural order of erasing tensor product symbols of $z_0\otimes\ldots\otimes z_p$, excluding those tensor product symbols that never get erased. The following example should clarify the point. Let $W = z_3z_0 \otimes z_1 \otimes z_2z_4$. This is a 2-cell of $Sym_*^{(4)}$. W is obtained from $z_0 \otimes z_1 \otimes z_2 \otimes z_3 \otimes z_4 = z_3 \otimes z_0 \otimes z_1 \otimes z_2 \otimes z_4$ by combining factors in some order. There are only 2 erasable tensor product symbols in this example. The natural order (last to first) corresponds to the chain, $z_3 \otimes z_0 \otimes z_1 \otimes z_2 \otimes z_4 \to z_3 \otimes z_0 \otimes z_1 \otimes z_2z_4 \to z_3z_0 \otimes z_1 \otimes z_2z_4$. So, this chain shows up in $\theta_*(W)$ with positive sign, whereas the chain $z_3 \otimes z_0 \otimes z_1 \otimes z_2 \otimes z_4 \to z_3z_0 \otimes z_1 \otimes z_2 \otimes z_4 \to z_3z_0 \otimes z_1 \otimes z_2z_4$ shows up with a negative sign.

Now, θ_* is easily seen to be a chain map $Sym_*^{(p)} \to k[N\mathcal{S}_p]/k[N\mathcal{S}'_p]$. Geometrically, θ_* has the effect of subdividing a cell-complex (defined with cubical cells) into a simplicial space, so θ_* is a homotopy-equivalence. □

Remark 81. As an example, consider $|N\mathcal{S}_2|$. There are 6 2-cells, each represented by a copy of I^2. The 2-cell labelled by the permutation $\{0, 1, 2\} \mapsto \{1, 0, 2\}$ consists of the chains $z_1 \otimes z_0 \otimes z_2 \to z_1 \otimes z_0z_2 \to z_1z_0z_2$ and $-(z_1\otimes z_0\otimes z_2 \to z_1z_0\otimes z_2 \to z_1z_0z_2)$. Hence, the boundary is the sum of 1-chains: $[(z_1\otimes z_0z_2 \to z_1z_0z_2)-(z_1\otimes z_0\otimes z_2 \to z_1z_0z_2)+(z_1\otimes z_0\otimes z_2 \to z_1\otimes z_0z_2)]-[(z_1z_0\otimes z_2 \to z_1z_0z_2)-(z_1\otimes z_0\otimes z_2 \to z_1z_0z_2)+(z_1\otimes z_0\otimes z_2 \to z_1z_0\otimes z_2)] = (z_1\otimes z_0z_2 \to z_1z_0z_2)+(z_1\otimes z_0\otimes z_2 \to z_1\otimes z_0z_2)-(z_1z_0\otimes z_2 \to z_1z_0z_2)-(z_1\otimes z_0\otimes z_2 \to z_1z_0\otimes z_2)$. This 1-chains correspond to the 4 edges of the square. Thus, in our example this 2-cell of $|N\mathcal{S}_p|$ will correspond to $z_1z_0z_2 \in Sym_2^{(2)}$, and its boundary in $|N\mathcal{S}_p/N\mathcal{S}'_p|$ will consist of the two edges adjacent to the vertex labeled$z_0 \otimes z_1 \otimes z_2$, with appropriate signs: $(z_0 \otimes z_1 \otimes z_2 \to z_1 \otimes z_0z_2) - (z_0 \otimes z_1 \otimes z_2 \to z_1z_0 \otimes z_2)$. The corresponding boundary in $Sym_1^{(2)}$ will be $(z_1 \otimes z_0z_2)-(z_1z_0 \otimes z_2)$, matching the boundary map already defined on $Sym_*^{(p)}$. See Figs. 3 and 4.

Now, with one piece of new notation, we may re-interpret Thm. 73.

Definition 82. Let G be a group. Let k_0 be the chain complex consisting of k concentrated in degree 0, with trivial G-action. If X_* is a right G-complex, Y_* is a left G-complex with $k_0 \hookrightarrow Y_*$ as a G-subcomplex, then define the *equivariant half-smash tensor product* of the two complexes:

$$X_* \circledS_G Y_* := (X_* \otimes_{kG} Y_*) \,/\, (X_* \otimes_{kG} k_0)$$

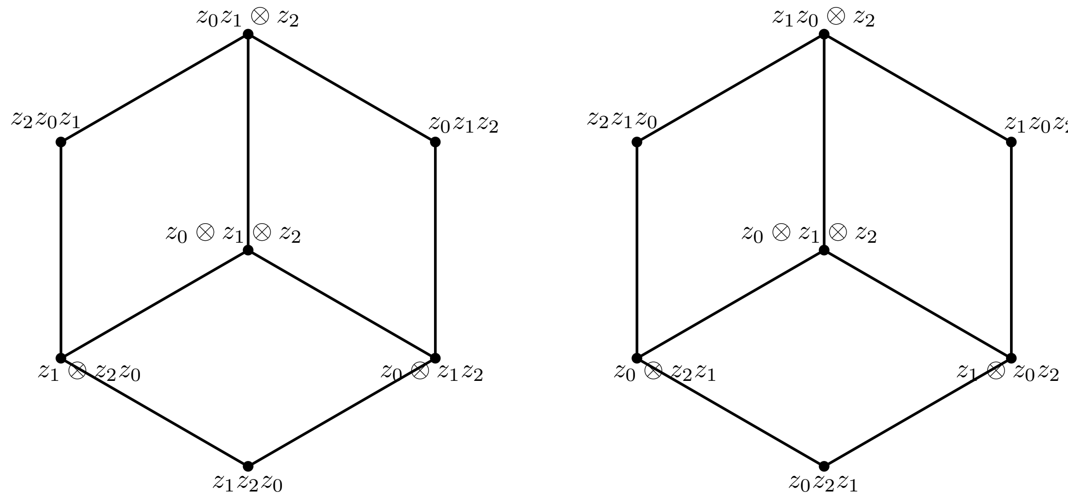

FIGURE 3. $|N\mathcal{S}_2|$ consists of six squares, grouped into two hexagons that share a common center vertex

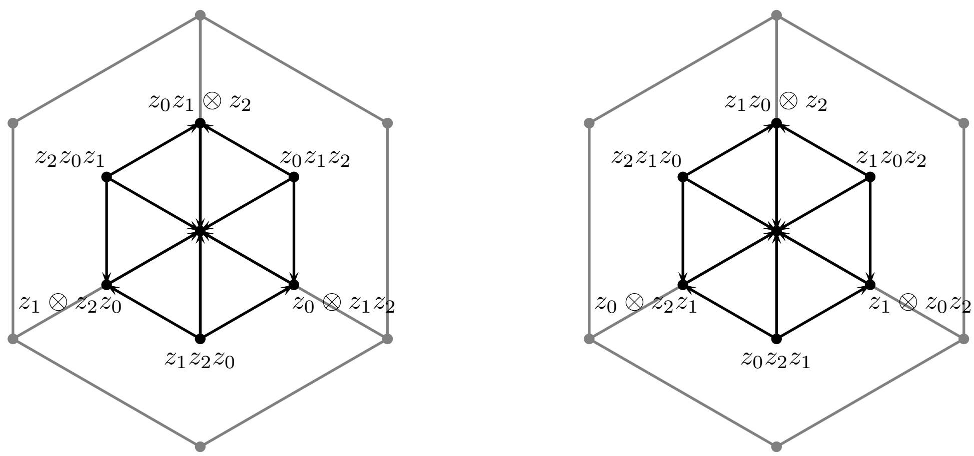

FIGURE 4. $Sym^{(2)} \simeq N\mathcal{S}_2/N\mathcal{S}_2'$. The center of each hexagon is $z_0 \otimes z_1 \otimes z_2$.

Corollary 83. *There is spectral sequence converging (weakly) to $\widetilde{H}S_*(A)$ with*

$$E^1_{p,q} \cong \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q}\left(E_*G_u \otimes_{G_u} Sym_*^{(p)}; k\right),$$

where G_u is the isotropy subgroup for the chosen representative of $u \in X^{p+1}/\Sigma_{p+1}$.

9. PROPERTIES OF THE COMPLEX $Sym_*^{(p)}$

9.1. **Algebra Structure of Sym_*.** We may consider $Sym_* := \bigoplus_{p \geq 0} Sym_*^{(p)}$ as a bigraded differential algebra, where $bideg(W) = (p+1, i)$ for $W \in Sym_i^{(p)}$. The product $\boxtimes : Sym_i^{(p)} \otimes Sym_j^{(q)} \to Sym_{i+j}^{(p+q+1)}$ is defined by: $W \boxtimes V := W \otimes V'$, where V' is obtained from V by replacing each formal indeterminate z_r by z_{r+p+1} for $0 \leq r \leq q$. Eq. 44 then implies:

$$d(W \boxtimes V) = d(W) \boxtimes V + (-1)^{bideg(W)_2} W \boxtimes d(V), \tag{46}$$

where $bideg(W)_2$ is the second component of $bideg(W)$.

Proposition 84. *The product $\boxtimes$ is defined on the level of homology. Furthermore, this product (on both the chain level and homology level) is skew commutative in a twisted sense: $W \boxtimes V = (-1)^{ij}\tau(V \boxtimes W)$, where $bideg(W) = (p+1, i)$, $bideg(V) = (q+1, j)$, and τ is the permutation sending $\{0, 1, \ldots, q, q+1, q+2, \ldots, p+q, p+q+1\} \mapsto \{p+1, p+2, \ldots, p+q+1, 0, 1, \ldots, p-1, p\}$.*

Proof. Eq. (46) implies the product passes to homology classes. Now, suppose $W = Y_0 \otimes Y_1 \otimes \ldots \otimes Y_{p-i} \in Sym_i^{(p)}$ and $V = Z_0 \otimes Z_1 \otimes \ldots \otimes Z_{q-j} \in Sym_j^{(p)}$.

$$V \boxtimes W = V \otimes W' = (-1)^{\alpha} W' \otimes V, \tag{47}$$

where W' is related to W by replacing each z_r by z_{r+q+1}. The exponent $\alpha = deg(V)deg(W) = ij$ arises from the relations in $Sym_{i+j}^{(p+q+1)}$. (The fact that $deg(V) = i$ and $deg(W) = j$ may be made clear by observing that the degree of a formal tensor product in $Sym_*^{(s)}$ is equal to the number of *cut points*, that is, the number of places where a tensor product symbol may be inserted.) Next, apply the block transformation τ to Eq. (47) to obtain $\tau(V \boxtimes W) = (-1)^{\alpha}\tau(W' \otimes V) = (-1)^{\alpha} W \otimes V' = (-1)^{\alpha} W \boxtimes V$, where V' is obtained by replacing z_r by z_{r+p+1} in V. $\square$

9.2. **Computer Calculations.** In principle, the homology of $Sym_*^{(p)}$ may be found by using a computer. In fact, we have the following results up to $p = 7$:

Theorem 85. *For $0 \leq p \leq 7$, the groups $H_*(Sym_*^{(p)})$ are free abelian and have Poincaré polynomials $P_p(t) := P\left(H_*(Sym_*^{(p)}); t\right)$:*

$$\begin{aligned} P_0(t) &= 1, \\ P_1(t) &= t, \\ P_2(t) &= t + 2t^2, \\ P_3(t) &= 7t^2 + 6t^3, \\ P_4(t) &= 43t^3 + 24t^4, \\ P_5(t) &= t^3 + 272t^4 + 120t^5, \\ P_6(t) &= 36t^4 + 1847t^5 + 720t^6, \\ P_7(t) &= 829t^5 + 13710t^6 + 5040t^7. \end{aligned}$$

Proof. These computations were performed using scripts written for the computer algebra systems `GAP` [9] and `Octave` [6]. $\square$

We conjecture that the $H_*(Sym_*^{(p)})$ is always free abelian.

9.3. **Representation Theory of $H_*(Sym_*^{(p)})$.** By remark 79, the groups $H_i(Sym_*^{(p)}; k)$ are $k\Sigma_{p+1}$-modules, so it seems natural to investigate the irreducible representations comprising these modules.

Proposition 86. *Let $C_{p+1} \hookrightarrow \Sigma_{p+1}$ be the cyclic group of order $p+1$, embedded into the symmetric group as the subgroup generated by the permutation $\tau_p := (0, p, p-1, \ldots, 1)$. Then there is a Σ_{p+1}-isomorphism: $H_p(Sym_*^{(p)}) \cong AC_{p+1} \uparrow \Sigma_{p+1}$, i.e., the alternating representation of the cyclic group, induced up to the symmetric group. Note, for p even, AC_{p+1} coincides with the trivial representation IC_{p+1}.*

Moreover, $H_p(Sym_*^{(p)})$ is generated by the elements $\sigma(b_p)$, for the distinct cosets σC_{p+1}, where $b_p := \sum_{j=0}^{p} (-1)^{jp}\tau_p^j(z_0 z_1 \ldots z_p)$.

Proof. Let w be a general element of $Sym_p^{(p)}$, $w = \sum_{\sigma \in \Sigma_{p+1}} c_\sigma \sigma(z_0 z_1 \ldots z_p)$, where c_σ are constants in k. $H_p(Sym_*^{(p)})$ consists of those w such that $d(w) = 0$. That is,

$$0 = \sum_{\sigma \in \Sigma_{p+1}} \sum_{i=0}^{p-1} (-1)^i c_\sigma \sigma(z_0 \ldots z_i \otimes z_{i+1} \ldots z_p). \tag{48}$$

Now for each σ, the terms corresponding to $\sigma(z_0 \ldots z_i \otimes z_{i+1} \ldots z_p)$ occur in pairs in the above formula. The obvious term of the pair is $(-1)^i c_\sigma \sigma(z_0 \ldots z_i \otimes z_{i+1} \ldots z_p)$. Not so obviously, the second term of the pair is $(-1)^{(p-i-1)i}(-1)^{p-i-1} c_\rho \rho(z_0 \ldots z_{p-i-1} \otimes z_{p-i} \ldots z_p)$, where $\rho = \sigma\tau_p^{p-i}$. Thus, if $d(w) = 0$,

then we must have $(-1)^i c_\sigma + (-1)^{(p-i-1)(i+1)} c_\rho = 0$, or $c_\rho = (-1)^{(p-i)(i+1)} c_\sigma$. Set $j = p - i$, so that $c_\rho = (-1)^{j(p-j+1)} c_\sigma = (-1)^{jp} c_\sigma$. This shows that the only restrictions on the coefficients c_σ are that the absolute values of coefficients corresponding to $\sigma, \sigma\tau_p, \sigma\tau_p^2, \ldots$ must be the same, and their corresponding signs in w alternate if and only if p is odd; otherwise, they have the same signs. Clearly, the elements $\sigma(b_p)$ for distinct cosets σC_{p+1} represents an independent set of generators over k for $H_p(Sym_*^{(p)})$.

Observe that b_p is invariant under the action of $sign(\tau_p)\tau_p$, and so b_p generates an alternating representation AC_{p+1} over k. Induced up to Σ_{p+1}, we obtain the representation $AC_{p+1} \uparrow \Sigma_{p+1}$ of dimension $(p+1)!/(p+1) = p!$, generated by the elements $\sigma(b_p)$ as in the proposition. □

Definition 87. For a given proper partition $\lambda = [\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_s]$ of the $p+1$ integers $\{0, 1, \ldots, p\}$, an element W of $Sym_*^{(p)}$ will designated as *type* λ if it equivalent to $\pm(Y_0 \otimes Y_1 \otimes Y_2 \otimes \ldots \otimes Y_s)$ with $deg(Y_i) = \lambda_i - 1$. That is, each Y_i has λ_i factors. The notation $Sym_\lambda^{(p)}$ or Sym_λ will denote the k-submodule of $Sym_{p-s}^{(p)}$ generated by all elements of type λ.

In what follows, $|\lambda|$ will refer to the number of components of λ. The action of Σ_{p+1} leaves Sym_λ invariant for any given λ, so the there is a decomposition as $k\Sigma_{p+1}$-module:

$$Sym_{p-s}^{(p)} = \bigoplus_{\lambda \vdash (p+1), |\lambda| = s+1} Sym_\lambda.$$

Proposition 88. *For a given proper partition $\lambda \vdash (p+1)$,*
(a) Sym_λ contains exactly one alternating representation $A\Sigma_{p+1}$ iff λ contains no repeated components.
(b) Sym_λ contains exactly one trivial representation $I\Sigma_{p+1}$ iff λ contains no repeated even components.

Proof. Sym_λ is a quotient of the regular representation, since it is the image of the Σ_{p+1}-map $\pi_\lambda : k\Sigma_{p+1} \to Sym_\lambda$, $\sigma \mapsto \psi_\lambda s$, where $s \in \Sigma_{p+1}^{\mathrm{op}}$ is the ΔS-automorphism of $[p]$ corresponding to σ and ψ_λ is a Δ-morphism $[p] \to [\,|\lambda|\,]$ that sends the points $0, \ldots, \lambda_0 - 1$ to 0, the points $\lambda_0, \ldots, \lambda_0 + \lambda_1 - 1$ to 1, and so on. Hence, there can be at most 1 copy of $A\Sigma_{p+1}$ and at most 1 copy of $I\Sigma_{p+1}$ in Sym_λ.

Let W be the "standard" element of Sym_λ. That is, the indeterminates z_i occur in W in numerical order and the degrees of monomials of W are in decreasing order. $A\Sigma_{p+1}$ exists in Sym_λ iff the element $V = \sum_{\sigma \in \Sigma_{p+1}} sign(\sigma)\sigma(W)$ is non-zero. Suppose that some component of λ is repeated, say $\lambda_i = \lambda_{i+1} = \ell$. If $W = Y_0 \otimes Y_1 \otimes \ldots \otimes Y_s$, then $deg(Y_i) = deg(Y_{i+1}) = \ell - 1$. Now, we know that $W = (-1)^{deg(Y_i)deg(Y_{i+1})} Y_0 \otimes \ldots \otimes Y_{i+1} \otimes Y_i \otimes \ldots Y_s = -(-1)^\ell \alpha(W)$, for the permutation $\alpha \in \Sigma_{p+1}$ that exchanges the indices of indeterminates in Y_i with those in Y_{i+1} in an order-preserving way. In V, the term $\alpha(W)$ shows up with sign $sign(\alpha) = (-1)^\ell$, thus cancelling with W. Hence, $V = 0$, and no alternating representation exists.

If, on the other hand, no component of λ is repeated, then no term $\pm\alpha(W)$ can be equivalent to W for $\alpha \neq \mathrm{id}$, so V survives as the generator of $A\Sigma_{p+1}$ in Sym_λ.

A similar analysis applies for trivial representations. This time, we examine $U = \sum_{\sigma \in \Sigma_{p+1}} \sigma(W)$, which would be a generator for $I\Sigma_{p+1}$ if it were non-zero. As before, if there is a repeated component, $\lambda_i = \lambda_{i+1} = \ell$, then $W = (-1)^{\ell-1}\alpha(W)$. However, this time, W cancels with $\alpha(W)$ only if $\ell - 1$ is odd. That is, $|\lambda_i| = |\lambda_{i+1}|$ is even. If $\ell - 1$ is even, or if all λ_i are distinct, then the element U must be non-zero. □

Proposition 89. *$H_i(Sym_*^{(p)})$ contains an alternating representation for each partition $\lambda \vdash (p+1)$ with $|\lambda| = p - i$ such that no component of λ is repeated.*

Proof. This proposition will follow from the fact that $d(V) = 0$ for any generator V of an alternating representation in Sym_λ. Then, by Schur's Lemma, the alternating representation must survive at the level of homology.

Let $V = \sum_{\sigma \in \Sigma_{p+1}} sign(\sigma)\sigma(W)$ be the generator mentioned in Prop. 88. $d(V)$ consists of individual terms $d_j(\sigma(W)) = \sigma(d_j(W))$ along with appropriate signs. For a given j, $d_j(W)$ is identical to W except at some monomial Y_i, where a tensor product symbol is inserted. We will introduce some notation to make the argument a little cleaner. If $Y = z_{i_0} z_{i_1} \ldots z_{i_r}$ is a monomial, then the notation $Y\{s, \ldots, t\}$ refers to the monomial $z_{i_s} z_{i_{s+1}} \ldots z_{i_t}$, assuming $0 \leq s \leq t \leq r$. Now, we write

$$d_j(W) = (-1)^{a+\ell} Y_0 \otimes \ldots \otimes Y_i\{0, \ldots, \ell\} \otimes Y_i\{\ell+1, \ldots, m\} \otimes \ldots \otimes Y_s, \tag{49}$$

where $a = deg(Y_0) + \ldots + deg(Y_{i-1})$. Use the relations in Sym_* to rewrite Eq. (49):

$$(-1)^{(a+\ell)+\ell(m-\ell-1)} Y_0 \otimes \ldots \otimes Y_i\{\ell+1, \ldots, m\} \otimes Y_i\{0, \ldots, \ell\} \otimes \ldots \otimes Y_s. \tag{50}$$

Let α be the permutation that relabels indices in such a way that $Y_i\{0, \ldots, m-\ell-1\} \mapsto Y_i\{\ell+1, \ldots, m\}$ and $Y_i\{m-\ell, \ldots, m\} \mapsto Y_i\{0, \ldots, \ell\}$, so that the following is equivalent to Eq. (50).

$$(-1)^{a+m\ell-\ell^2} \alpha\left(Y_0 \otimes \ldots \otimes Y_i\{0, \ldots, m-\ell-1\} \otimes Y_i\{m-\ell, \ldots, m\} \otimes \ldots \otimes Y_s\right) \tag{51}$$

Now, Eq.(51) also occurs in $d_{j'}\left(sign(\alpha)\alpha(W)\right)$ for some j'. This term looks like:

$$sign(\alpha)(-1)^{a+m-\ell-1} \alpha\big(Y_0 \otimes \ldots \otimes Y_i\{0, \ldots, m-\ell-1\} \otimes Y_i\{m-\ell, \ldots, m\} \otimes \ldots \otimes Y_s\big) \tag{52}$$

$$= (-1)^{m\ell-\ell^2+a-1} \alpha\big(Y_0 \otimes \ldots \otimes Y_i\{0, \ldots, m-\ell-1\} \otimes Y_i\{m-\ell, \ldots, m\} \otimes \ldots \otimes Y_s\big) \tag{53}$$

Comparing the signs of Eq. (53) and Eq. (51), we verify the two terms canel each other out in the sum $d(V)$. □

By Proposition 89, it is clear that if $p+1$ is a triangular number – *i.e.*, $p+1$ is of the form $r(r+1)/2$ for some positive integer r, then the lowest dimension in which an alternating representation may occur is $p+1-r$, corresponding to the partition $\lambda = [r, r-1, \ldots, 2, 1]$. A little algebra yields the following statement for any p.

Corollary 90. *$H_i(Sym_*^{(p)})$ contains an alternating representation in degree $p+1-r$, where $r = \lfloor \sqrt{2p+9/4} - 1/2 \rfloor$. Moreover, there are no alternating representations present for $i \leq p-r$.*

There is not much known about the other irreducible representations occurring in the homology groups of $Sym_*^{(p)}$, however computational evidence shows that $H_i(Sym_*^{(p)})$ contains no trivial representation, $I\Sigma_{p+1}$, for $i \leq p-r$ (r as in the conjecture above) up to $p = 50$.

9.4. **Connectivity of $Sym_*^{(p)}$.** Quite recently, Vrećica and Živaljević [25] observed that the complex $Sym_*^{(p)}$ is isomorphic to the suspension of the cycle-free chessboard complex Ω_{p+1} (in fact, the isomorphism takes the form $k\left[S\Omega_{p+1}^+\right] \to Sym_*^{(p)}$, where Ω_{p+1}^+ is the augmented complex).

The m-chains of the complex Ω_n are generated by ordered lists, $L = \{(i_0, j_0), (i_1, j_1), \ldots, (i_m, j_m)\}$, where $1 \leq i_0 < i_1 < \ldots < i_m \leq n$, all $1 \leq j_s \leq n$ are distinct integers, and the list L is *cycle-free*. It may be easier to say what it means for L not to be cycle free: L is not cycle-free if there exists a subset $L_c \subseteq L$ and ordering of L_c so that $L_c = \{(\ell_0, \ell_1), (\ell_1, \ell_2), \ldots, (\ell_{t-1}, \ell_t), (\ell_t, \ell_0)\}$. The differential of Ω_n is defined on generators by:

$$d\left(\{(i_0, j_0), \ldots, (i_m, j_m)\}\right) := \sum_{s=0}^{m} (-1)^s \{(i_0, j_0), \ldots, (i_{s-1}, j_{s-1}), (i_{s+1}, j_{s+1}), \ldots, (i_m, j_m)\}.$$

For completeness, an explicit isomorphism shall be provided:

Proposition 91. *Let Ω_n^+ denote the augmented cycle-free $(n \times n)$-chessboard complex, where the unique (-1)-chain is represented by the empty $n \times n$ chessboard, and the boundary map on 0-chains takes a vertex to the unique (-1)-chain. For each $p \geq 0$, there is a chain isomorphism, $\omega_* : k\left[S\Omega_{p+1}^+\right] \to Sym_*^{(p)}$.*

Proof. Note that we may define the generating m-chains of $k\left[\Omega_{p+1}\right]$ as cycle-free lists L, with no requirement on the order of L, under the equivalence relation: $\sigma.L := \left\{(i_{\sigma^{-1}(0)}, j_{\sigma^{-1}(0)}), \ldots, (i_{\sigma^{-1}(m)}, j_{\sigma^{-1}(m)})\right\} \approx sign(\sigma)L$, for $\sigma \in \Sigma_{m+1}$. Suppose L is an $(m+1)$-chain of $S\Omega_{p+1}^+$ (*i.e.* an m-chain of Ω_{p+1}^+). Call a subset $L' \subseteq L$ a *queue* if there is a reordering of L' such that $L' = \{(\ell_0, \ell_1), (\ell_1, \ell_2), \ldots, (\ell_{t-1}, \ell_t)\}$. L' is called a *maximal queue* if it is not properly contained in any other queue. Since L is supposed to be cycle-free, we can partition L into some number of maximal queues, $L'_1, L'_2, \ldots, L'_q$. Let σ be a permutation representing the reordering of L into maximal ordered queues.

Now, each maximal ordered queue L'_i will correspond to a monomial of formal indeterminates z_i as follows.

$$L'_s := \{(\ell_0, \ell_1), (\ell_1, \ell_2), \ldots, (\ell_{t-1}, \ell_t)\} \mapsto z_{\ell_0 - 1} z_{\ell_1 - 1} \cdots z_{\ell_t - 1}. \tag{54}$$

For each maximal ordered queue, L'_s, denote the monomial obtained by formula (54) by Z_s. Let $k_1, k_2, \ldots, k_u$ be the numbers in $\{0, 1, 2, \ldots, p\}$ such that $k_r + 1$ does not appear in any pair $(i_s, j_s) \in L$. Now we may define ω_* on $L = L'_1 \cup L'_2 \cup \ldots \cup L'_q$.

$$\omega_{m+1}(L) := Z_1 \otimes Z_2 \otimes \ldots \otimes Z_q \otimes z_{k_1} \otimes z_{k_2} \otimes \ldots \otimes z_{k_u}. \tag{55}$$

Observe, if $L = \emptyset$ is the (-1)-chain of Ω^+_{p+1}, then there are no maximal queues in L, and so $\omega_0(\emptyset) = z_0 \otimes z_1 \otimes \ldots \otimes z_p$.

ω_* is a (well-defined) chain map with inverse given by essentially reversing the process. To each monomial $Z = z_{i_0} z_{i_1} \cdots z_{i_t}$ with $t > 0$, there is an associated ordered queue $L' = \{(i_0+1, i_1+1), (i_1+1, i_2+1), \ldots (i_{t-1}+1, i_t+1)\}$. If the monomial is a singleton, $Z = z_{i_0}$, the associated ordered queue will be the empty set. Now, given a generator $Z_1 \otimes Z_2 \otimes \ldots \otimes Z_q \in Sym_*^{(p)}$, map it to the list $L := L'_1 \cup L'_2 \cup \ldots \cup L'_q$, preserving the original order of indices. □

Theorem 92. *$Sym_*^{(p)}$ is $\lfloor \frac{2}{3}(p-1) \rfloor$-connected.*

Proof. See Thm. 10 of [25]. □

This remarkable fact yields the following useful corollaries:

Corollary 93. *The spectral sequences of Thm. 73 and Cor. 83 converge strongly to $\widetilde{H}S_*(A)$.*

Proof. This relies on the fact that the connectivity of the complexes $Sym_*^{(p)}$ is a non-decreasing function of p. Fix $n \geq 0$, and consider $\bigoplus_u H_n\left(E_*G_u \otimes_{G_u} Sym_*^{(p)}\right)$, the component of E^1 residing at position p, q for $p + q = n$. A priori, the induced differentials whose sources are $E^1_{p,q}, E^2_{p,q}, E^3_{p,q}, \ldots$ will have as targets certain subquotients of $\bigoplus_u H_{n-1}\left(E_*G_u \otimes_{G_u} Sym_*^{(p+1)}\right)$, $\bigoplus_u H_{n-1}\left(E_*G_u \otimes_{G_u} Sym_*^{(p+2)}\right)$, $\bigoplus_u H_{n-1}\left(E_*G_u \otimes_{G_u} Sym_*^{(p+3)}\right)$, etc. Now, if $n - 1 < \lfloor (2/3)(p+k-1) \rfloor$ for some $k \geq 0$, then for $K > k$, we have $H_{n-1}\left(Sym_*^{(p+K)}\right) = 0$, hence also, $H_{n-1}\left(E_*G_u \otimes_{G_u} Sym_*^{(p+K)}\right) = 0$, using the fibration mentioned in the proof of Thm. 73 and the Hurewicz Theorem. Thus, the induced differential d^k is zero for all $k \geq K$.

On the other hand, the induced differentials whose targets are $E^1_{p,q}, E^2_{p,q}, E^3_{p,q}, \ldots$ must be zero after stage p, since there are no non-zero components with $p < 0$. □

Corollary 94. *For each $i \geq 0$, there is a positive integer N_i so that if $p \geq N_i$, there is an isomorphism $H_i\left(\mathscr{G}_p \widetilde{\mathscr{Y}_*}\right) \cong \widetilde{H}S_i(A)$.*

Corollary 95. *If A is finitely-generated over a Noetherian ground ring k, then $HS_*(A)$ is finitely-generated over k in each degree.*

Proof. Examination of the E^1 term shows that the n^{th} reduced symmetric homology group of A is a subquotient of $\bigoplus_{p \geq 0} \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_n\left(E_*G_u \otimes_{G_u} Sym_*^{(p)}; k\right)$. Each $H_n\left(E_*G_u \otimes_{G_u} Sym_*^{(p)}; k\right)$ is a finite-dimensional k-module. The inner sum is finite as long as X is finite. Thm. 92 shows the outer sum is finite as well. □

The bounds on connectivity are conjectured to be tight. This is certainly true for $p \equiv 1 \pmod 3$, based on Thm. 16 of [25]. Corollary 12 of the same paper establishes that either $H_{2k}\left(Sym_*^{(3k-1)}\right) \neq 0$ or $H_{2k}\left(Sym_*^{(3k)}\right) \neq 0$. For $k \leq 2$, both statements are true. When the latter condition is true, this gives a tight bound on connectivity for $p \equiv 0 \pmod 3$. When the former is true, there is not enough information for a tight bound, since we are more interested in proving that $H_{2k-1}\left(Sym_*^{(3k-1)}\right)$ is non-zero, since for $k = 1, 2$, we have computed the integral homology, $H_1\left(Sym_*^{(2)}\right) = \mathbb{Z}$ and $H_3\left(Sym_*^{(5)}\right) = \mathbb{Z}$.

9.5. **Filtering $Sym_*^{(p)}$ by partition types.** In 9.3, we saw that $Sym_*^{(n)}$ decomposes over $k\Sigma_{n+1}$ as a direct sum of the submodules Sym_λ for partitions $\lambda \vdash (n+1)$. We may filter $Sym_*^{(n)}$ by the size of the largest component of the partition,

$$\mathscr{F}_p Sym_q^{(n)} := \bigoplus_{\lambda \vdash (n+1), |\lambda| = n+1-(p+q), \lambda_0 \leq p+1} Sym_\lambda,$$

where $\lambda = [\lambda_0, \lambda_1, \ldots, \lambda_{n-q}]$, is written in non-increasing order. The differential of $Sym_*^{(n)}$ respects this filtering, since it can only reduce the size of partition components. With respect to this filtering, we have an E^0 term for a spectral sequence:

$$E^0_{p,q} \cong \bigoplus_{\lambda \vdash (n+1), |\lambda| = n+1-(p+q), \lambda_0 = p+1} Sym_\lambda.$$

The vertical differential d^0 is induced from d by keeping only those terms of $d(W)$ that share largest component size with W.

10. A Partial Resolution

As before, k is a commutative ground ring. In this section, we find an explicit partial resolution of the trivial ΔS^{op}-module $\underline{k}$ by projective modules, allowing the computation of $HS_0(A)$ and $HS_1(A)$ for a unital associative k-algebra A. The resolution will be constructed through a number of technical lemmas. Recall from 1.4, the modules $k\left[\mathrm{Mor}_{\Delta S}\left(-, [q]\right)\right]$ are projective as ΔS^{op}-module. In proving exactness, it suffices to examine the individual sub-k-modules, $k\left[\mathrm{Mor}_{\Delta S}\left([n], [q]\right)\right]$.

Lemma 96. *For each $n \geq 0$, the sequence, $0 \leftarrow k \xleftarrow{\epsilon} k\left[\mathrm{Mor}_{\Delta S}\left([n], [0]\right)\right] \xleftarrow{\rho} k\left[\mathrm{Mor}_{\Delta S}\left([n], [2]\right)\right]$ is exact, where ϵ is defined by $\epsilon(\phi) = 1$ for any morphism $\phi : [n] \to [0]$, and ρ is defined by $\rho(\psi) = (x_0x_1x_2) \circ \psi - (x_2x_1x_0) \circ \psi$ for any morphism $\psi : [n] \to [2]$. Note, $x_0x_1x_2$ and $x_2x_1x_0$ are ΔS morphisms $[2] \to [0]$ written in tensor notation.*

Proof. Clearly, ϵ is surjective. Now, $\epsilon\rho = 0$, since $\rho(\psi)$ consists of two morphisms with opposite signs. Let $\phi_0 = x_0x_1 \ldots x_n : [n] \to [0]$. The kernel of ϵ is spanned by elements $\phi - \phi_0$ for $\phi \in \mathrm{Mor}_{\Delta S}([n], [0])$. So, it suffices to show that the submodule of $k\left[\mathrm{Mor}_{\Delta S}\left([n], [0]\right)\right]$ generated by $(x_0x_1x_2)\psi - (x_2x_1x_0)\psi$ for $\psi : [n] \to [2]$ contains all of the elements $\phi - \phi_0$. In other words, it suffices to find a sequence $\phi =: \phi_k, \phi_{k-1}, \ldots, \phi_2, \phi_1, \phi_0$ so that each ϕ_i is obtained from ϕ_{i+1} by reversing the order of 3 (possibly emtpy) blocks, $XYZ \to ZYX$. Let $\phi = x_{i_0}x_{i_1} \ldots x_{i_n}$. If $\phi = \phi_0$, we may stop here. Otherwise, we may produce a sequence ending in ϕ_0 by way of a certain family of rearrangements:

k-rearrangment: $x_{i_0}x_{i_1} \ldots x_{i_{k-1}}x_{i_k}x_{k+1} \ldots x_n \rightsquigarrow x_{k+1} \ldots x_n x_{i_k} x_{i_0} x_{i_1} \ldots x_{i_{k-1}}$, where $i_k \neq k$. That is, a k-rearrangement only applies to those monomials that agree with ϕ_0 in the final $n-k$ indeterminates, but not in the final $n-k+1$ indeterminates. If $k = n$, then this rearrangement reduces to the cyclic rearrangment, $x_{i_0}x_{i_1} \ldots x_{i_n} \rightsquigarrow x_{i_n}x_{i_0}x_{i_1} \ldots x_{i_{n-1}}$.

Beginning with ϕ, perform n-rearrangements until the final indeterminate is x_n. For convenience of notation, let this new monomial be $x_{j_0}x_{j_1} \ldots x_{j_n}$. (Of course, $j_n = n$.) If $j_k = k$ for all $k = 0, 1, \ldots, n$, then we are done. Otherwise, there will be a number k such that $j_k \neq k$ but $j_{k+1} = k+1, \ldots, j_n = n$. Perform a k-rearrangement followed by enough n-rearrangements so that the final indeterminate is again x_n. The net result of these rearrangements is that the ending block $x_{k+1}x_{k+2} \ldots x_n$ remains fixed while the beginning block $x_{j_0}x_{j_1} \ldots x_{j_k}$ becomes cyclically permuted to $x_{j_k}x_{j_0} \ldots x_{j_{k-1}}$. It is clear that applying this combination of rearrangements repeatedly will finally obtain a monomial $x_{\ell_0}x_{\ell_1} \ldots x_{\ell_{k-1}}x_kx_{k+1} \ldots x_n$. Now repeat the process, until after a finite number of steps, we finally obtain ϕ_0. $\square$

Let $\mathscr{B}_n := \{x_{i_0}x_{i_1} \ldots x_{i_{k-1}} \otimes x_{i_k} \otimes x_{k+1}x_{k+2} \ldots x_n \ : \ 1 \leq k \leq n, i_k \neq k\}$. $k[\mathscr{B}_n]$ is a free submodule of $k\left[\mathrm{Mor}_{\Delta S}\left([n], [2]\right)\right]$ of size $(n+1)! - 1$.

Corollary 97. *When restricted to $k[\mathscr{B}_n]$, the map ρ of Lemma 96 is surjective onto the kernel of ϵ.*

Proof. In the proof of Lemma 96, the n-rearrangements correspond to the image of elements $x_{i_0} \ldots x_{i_{n-1}} \otimes x_{i_n} \otimes 1$, with $i_n \neq n$. For $k < n$, k-rearrangements correspond to the image of elements $x_{i_0} \ldots x_{i_{k-1}} \otimes x_{i_k} \otimes x_{k+1} \ldots x_n$, with $i_k \neq k$. $\square$

Lemma 98. *$k\left[\mathrm{Mor}_{\Delta S}\left([n], [m]\right)\right]$ is a free k-module of rank $(m+n+1)!/m!$.*

Proof. A morphism $\phi : [n] \to [m]$ of ΔS is nothing more than an assignment of $n+1$ objects into $m+1$ compartments, along with a total ordering of the original $n+1$ objects, hence $\#\mathrm{Mor}_{\Delta S}([n],[m]) = \binom{m+n+1}{m}(n+1)! = \frac{(m+n+1)!}{m!}$. $\square$

Lemma 99. *$\rho|_{k[\mathscr{B}_n]}$ is an isomorphism $k[\mathscr{B}_n] \cong \ker \epsilon$.*

Proof. Since the rank of $k\left[\mathrm{Mor}_{\Delta S}([n],[0])\right]$ is $(n+1)!$, the rank of the $\ker\epsilon$ is $(n+1)!-1$. The isomorphism then follows from Corollary 97. $\square$

Lemma 100. *The relations of the form:*

$$XY \otimes Z \otimes W + W \otimes ZX \otimes Y + YZX \otimes 1 \otimes W + W \otimes YZ \otimes X \approx 0 \tag{56}$$

$$\text{and} \qquad 1 \otimes X \otimes 1 \approx 0 \tag{57}$$

collapse $k\big[\mathrm{Mor}_{\Delta S}([n],[2])\big]$ onto $k[\mathscr{B}_n]$.

Proof. This proof proceeds in multiple steps.

Step 1. [Degeneracy Relations] $X \otimes Y \otimes 1 \approx X \otimes 1 \otimes Y \approx 1 \otimes X \otimes Y$.

First, observe that letting $X = Y = W = 1$ in Eq. (56) yields $Z \otimes 1 \otimes 1 \approx 0$, since $1 \otimes Z \otimes 1 \approx 0$. Then, letting $X = Z = W = 1$ in Eq. (56) produces $1 \otimes 1 \otimes Y \approx 0$. Thus, any formal tensor with two trivial factors is equivalent to 0. Next, let $Z = W = 1$ in Eq. (56). Then using the above observation, we obtain $1 \otimes X \otimes Y + 1 \otimes Y \otimes X \approx 0$, that is, $1 \otimes X \otimes Y \approx -(1 \otimes Y \otimes X)$. Then, if we let $X = W = 1$, we obtain $Y \otimes Z \otimes 1 + 1 \otimes Z \otimes Y \approx 0$, which is equivalent to $Y \otimes Z \otimes 1 - 1 \otimes Y \otimes Z \approx 0$. Finally, let $X = Y = 1$ in Eq. (56). The expression reduces to $Z \otimes 1 \otimes W - 1 \otimes Z \otimes W \approx 0$.

Step 2. [Sign Relation] $X \otimes Y \otimes Z \approx -(Z \otimes Y \otimes X)$.

Let $Y = 1$ in Eq. (56), and use the degeneracy relations to rewrite the result as $X \otimes Z \otimes W + 1 \otimes W \otimes ZX + 1 \otimes ZX \otimes W + W \otimes Z \otimes X \approx 0$. Since $1 \otimes ZX \otimes W \approx -(1 \otimes W \otimes ZW)$, the desired result follows: $X \otimes Z \otimes W + W \otimes Z \otimes X \approx 0$.

Step 3. [Hochschild Relation] $XY \otimes Z \otimes 1 - X \otimes YZ \otimes 1 + ZX \otimes Y \otimes 1 \approx 0$. This relation is named after the similar relation, $XY \otimes Z - X \otimes YZ + ZX \otimes Y$, that arises in the Hochschild complex.

Let $W = 1$ in Eq. (56), and use the degeneracy and sign relations to obtain the desired result.

Step 4. [Cyclic Relation] $\displaystyle\sum_{j=0}^{n} \tau_n^j \left(x_{i_0}x_{i_1}\dots x_{i_{n-1}} \otimes x_{i_n} \otimes 1\right) \approx 0$, where $\tau_n \in \Sigma_{n+1}$ is the $(n+1)$-cycle $(0, n, n-1, \dots, 2, 1)$, which acts by permuting the indices.

For $n = 0$, there are no such relations (indeed, no relations at all). For $n = 1$, the cyclic relation takes the form $x_0 \otimes x_1 \otimes 1 + x_1 \otimes x_0 \otimes 1 \approx 0$, which follows from degeneracy and sign relations.

Assume now that $n \geq 2$. For each $k = 1, 2, \dots, n-1$, define:

$$\begin{cases} A_k := & x_{i_0}x_{i_1}\dots x_{i_{k-1}}, \\ B_k := & x_{i_k}, \\ C_k := & x_{i_{k+1}}\dots x_{i_n}. \end{cases}$$

By the Hochschild relation, $0 \approx \sum_{k=1}^{n-1}(A_kB_k \otimes C_k \otimes 1 - A_k \otimes B_kC_k \otimes 1 + C_kA_k \otimes B_k \otimes 1)$. But for $k \leq n-2$, $A_kB_k \otimes C_k \otimes 1 = A_{k+1} \otimes B_{k+1}C_{k+1} \otimes 1$. Thus, after some cancellation,

$$0 \approx -A_1 \otimes B_1C_1 \otimes 1 + A_{n-1}B_{n-1} \otimes C_{n-1} \otimes 1 + \sum_{k=1}^{n-1} C_kA_k \otimes B_k \otimes 1. \tag{58}$$

Now observe that sign and degeneracy relations imply that $-A_1 \otimes B_1C_1 \otimes 1 \approx x_{i_1}\dots x_{i_n} \otimes x_{i_0} \otimes 1$. The term $A_{n-1}B_{n-1} \otimes C_{n-1} \otimes 1$ is equal to $x_{i_0}\dots x_{i_{n-1}} \otimes x_{i_n} \otimes 1$, and for $1 \leq k \leq n-1$, $C_kA_k \otimes B_k \otimes 1 = x_{i_{k+1}}\dots x_{i_n}x_{i_0}\dots x_{i_{k-1}} \otimes x_{i_k} \otimes 1$. Thus, Eq. (58) can be rewritten as the cyclic relation,

$$0 \approx (x_{i_0}\dots x_{i_{n-1}} \otimes x_{i_n} \otimes 1) + \sum_{k=0}^{n-1} x_{i_{k+1}}\dots x_{i_n}x_{i_0}\dots x_{i_{k-1}} \otimes x_{i_k} \otimes 1.$$

Step 5. Every element of the form $X \otimes Y \otimes 1$ is equivalent to a linear combination of elements of $\mathscr{B}_n$.

To prove this, we shall induct on the size of Y. Suppose Y consists of a single indeterminate. That is, $X \otimes Y \otimes 1 = x_{i_0}\dots x_{i_{n-1}} \otimes x_{i_n} \otimes 1$. Now, if $i_n \neq n$, we are done. Otherwise, we use the cyclic relation to write $x_{i_0}\dots x_{i_{n-1}} \otimes x_{i_n} \otimes 1 \approx -\sum_{j=1}^{n} \tau_n^j \left(x_{i_0}\dots x_{i_{n-1}} \otimes x_{i_n} \otimes 1\right)$.

Now suppose $k \geq 1$ and any element $Z \otimes W \otimes 1$ with $|W| = k$ is equivalent to an element of $k[\mathscr{B}_n]$. Consider $X \otimes Y \otimes 1 = x_{i_0} \dots x_{i_{n-k-1}} \otimes x_{i_{n-k}} \dots x_{i_n} \otimes 1$. Let

$$\begin{cases} A := & x_{i_0} x_{i_1} \dots x_{i_{n-k-1}}, \\ B := & x_{i_{n-k}} \dots x_{i_{n-1}}, \\ C := & x_{i_n}. \end{cases}$$

Then, by the Hochschild relation, $X \otimes Y \otimes 1 = A \otimes BC \otimes 1 \approx AB \otimes C \otimes 1 + CA \otimes B \otimes 1$. But since C has one indeterminate and B has k indeterminates, this last expression is equivalent to an element of $k[\mathscr{B}_n]$ by inductive hypothesis.

Step 6. [Modified Hochschild Relation] $XY \otimes Z \otimes W - X \otimes YZ \otimes W + ZX \otimes Y \otimes W \approx 0$, modulo $k[\mathscr{B}_n]$.

First, we show that $X \otimes Y \otimes W + Y \otimes X \otimes W \approx 0 \pmod{k[\mathscr{B}_n]}$. Indeed, if we let $Z = 1$ in Eq. (56), then sign and degeneracy relations yield: $X \otimes Y \otimes W + Y \otimes X \otimes W \approx XY \otimes W \otimes 1 + YX \otimes W \otimes 1$, *i.e.*, by step 5,

$$X \otimes Y \otimes W \approx -(Y \otimes X \otimes W) \pmod{k[\mathscr{B}_n]}. \tag{59}$$

Now, using sign and degeneracy relations, Eq. (56) can be re-expressed:

$$XY \otimes Z \otimes W - Y \otimes ZX \otimes W + YZX \otimes W \otimes 1 - X \otimes YZ \otimes W \approx 0.$$

Using Eq. (59), we then arrive at the modified Hochschild relation,

$$XY \otimes Z \otimes W - X \otimes YZ \otimes W + ZX \otimes Y \otimes W \approx 0 \pmod{k[\mathscr{B}_n]}. \tag{60}$$

Step 7. [Modified Cyclic Relation] $\displaystyle\sum_{j=0}^{k} \tau_k^j \left(x_{i_0} x_{i_1} \dots x_{i_{k-1}} \otimes x_{i_k} \otimes x_{i_{k+1}} \dots x_{i_n}\right) \approx 0$, modulo $k[\mathscr{B}_n]$.

Note, the $(k+1)$-cycle τ_k permutes the indices $i_0, i_1, \dots, i_k$, and fixes the rest.

Eq. (59) proves the modified cyclic relations for $k = 1$. The modified cyclic relation for $k \geq 2$ follows from the modified Hochschild relation in the same manner as in step 4. Of course, this time all equivalences are taken modulo $k[\mathscr{B}_n]$.

Step 8. Every element of the form $X \otimes Y \otimes x_n$ is equivalent to an element of $k[\mathscr{B}_n]$.

We shall use the modified cyclic and modified Hochschild relations in a similar way as cyclic and Hochschild relations were used in step 5. Again we induct on the size of Y. If $|Y| = 1$, then $X \otimes Y \otimes x_n = x_{i_0} \dots x_{i_{n-2}} \otimes x_{i_{n-1}} \otimes x_n$. If $i_{n-1} \neq n-1$, then we are done. Otherwise, use the modified cyclic relation to re-express $X \otimes Y \otimes x_n$ as a sum of elements of $k[\mathscr{B}_n]$.

Next, suppose $k \geq 1$ and any element $Z \otimes W \otimes x_n$ with $|W| = k$ is equivalent to an element of $k[\mathscr{B}_n]$. Consider an element $X \otimes Y \otimes x_n$ with $|Y| = k+1$. Write $Y = BC$ with $|B| = k$ and $|C| = 1$ and use the modified Hochschild relation to rewrite $X \otimes BC \otimes x_n$ in terms of two elements whose middle tensor factors are either B or C (modulo $k[\mathscr{B}_n]$). By inductive hypothesis, the rewritten expression must lie in $k[\mathscr{B}_n]$.

Step 9. Every element of $k[\mathrm{Mor}_{\Delta S}([n],[2])]$ is equivalent to a linear combination of elements from the following set:

$$\mathscr{C}_n := \{X \otimes x_{i_n} \otimes 1 \mid i_n \neq n\} \cup \{X \otimes x_{i_{n-1}} \otimes x_n \mid i_{n-1} \neq n-1\} \cup \{X \otimes Y \otimes Zx_n \mid |Z| \geq 1\} \tag{61}$$

Note, the k-module generated by $\mathscr{C}_n$ contains $k[\mathscr{B}_n]$.

Let $X \otimes Y \otimes Z$ be an arbitrary element of $k[\mathrm{Mor}_{\Delta S}([n],[2])]$. If $|X| = 0$, $|Y| = 0$, or $|Z| = 0$, then the degeneracy relations and step 5 imply that $X \otimes Y \otimes Z$ is equivalent to an element of $k[\mathscr{B}_n]$.

Suppose now that $|X|, |Y|, |Z| \geq 1$. If x_n occurs in X, use the relation $X \otimes Y \otimes W \approx -(Y \otimes X \otimes W)$ $\pmod{k[\mathscr{B}_n]}$ to ensure that x_n occurs in the middle factor. If x_n occurs in Z, use the sign relation and the above relation to put x_n into the middle factor. In any case, it suffices to assume our element has the form: $X \otimes Ux_nV \otimes Z$. Using the modified Hochschild relation, $X \otimes Ux_nV \otimes Z \approx -(Z \otimes V \otimes XUx_n) + Z \otimes VX \otimes Ux_n$, $\pmod{k[\mathscr{B}_n]}$. The first term is certainly in $k[\mathscr{C}_n]$, since $|X| \geq 1$. If $|U| > 0$, the second term also lies in $k[\mathscr{C}_n]$. If, on the other hand, $|U| = 0$, then step 8 implies that $Z \otimes VX \otimes x_n$ is an element of $k[\mathscr{B}_n]$.

Observe that Step 9 proves Lemma 100 for $n = 0, 1, 2$, since in these cases, any elements that fall within the set $\{X \otimes Y \otimes Zx_n \mid |Z| \geq 1\}$ must have either $|X| = 0$ or $|Y| = 0$, hence are equivalent via the degeneracy relation to elements of $k[\{X \otimes x_{i_n} \otimes 1 \mid i_n \neq n\}]$. In what follows, assume $n \geq 3$.

Step 10. Every element of $k\,[\mathrm{Mor}_{\Delta S}\,([n],[2])]$ is equivalent, modulo $k\,[\mathscr{B}_n]$, to a linear combination of elements from the following set:

$$\mathscr{D}_n := \{X \otimes x_{i_{n-2}} \otimes x_{n-1}x_n \mid i_{n-2} \neq n-2\} \cup \{X \otimes Y \otimes Zx_{n-1}x_n \mid |Z| \geq 1\}. \tag{62}$$

First, we require a relation that transports x_n from the end of a tensor:

$$W \otimes Z \otimes Xx_n \approx W \otimes x_nZ \otimes X \pmod{k\,[\mathscr{B}_n]}. \tag{63}$$

Letting $Y = x_n$ in Eq. (56), and making use of the sign relation, we have: $W \otimes Z \otimes Xx_n \approx W \otimes ZX \otimes x_n + x_nZX \otimes W \otimes 1 + W \otimes x_nZ \otimes X$. By steps 5 and 8, $W \otimes Z \otimes Xx_n \approx W \otimes x_nZ \otimes X$, modulo elements of $k\,[\mathscr{B}_n]$.

Now, let $X \otimes Y \otimes Z$ be an arbitrary element of $k\,[\mathrm{Mor}_{\Delta S}\,([n],[2])]$. Locate x_{n-1} and use the techniques of Step 9 to re-express $X \otimes Y \otimes Z$ as a linear combination of terms of the form: $X_j \otimes Y_j \otimes Z_jx_{n-1}$, modulo $k\,[\mathscr{B}_n]$. Our goal is to re-express each term as a linear combination of vectors in which x_n occurs only in the second tensor factor.

If x_n occurs in X_j, then observe $X_j \otimes Y_j \otimes Z_jx_{n-1} \approx -(Y_j \otimes X_j \otimes Z_jx_{n-1})$, $\pmod{k\,[\mathscr{B}_n]}$.

If x_n occurs in Z_j, then first substitute $Y = x_{n-1}$ into Eq. (56), obtaining the relation:

$$Xx_{n-1} \otimes Z \otimes W + W \otimes ZX \otimes x_{n-1} + x_{n-1}ZX \otimes 1 \otimes W + W \otimes x_{n-1}Z \otimes X \approx 0$$

$$\Rightarrow\ W \otimes Z \otimes Xx_{n-1} \approx W \otimes ZX \otimes x_{n-1} + W \otimes x_{n-1}Z \otimes X \pmod{k\,[\mathscr{B}_n]}$$

By the modified Hochschild relation, $W \otimes x_{n-1}Z \otimes X \approx Wx_{n-1} \otimes Z \otimes X + ZW \otimes x_{n-1} \otimes X$, $\pmod{k\,[\mathscr{B}_n]}$, then using sign relations etc., we obtain:

$$W \otimes Z \otimes Xx_{n-1} \approx W \otimes ZX \otimes x_{n-1} + Z \otimes X \otimes Wx_{n-1} - ZW \otimes X \otimes x_{n-1}, \pmod{k\,[\mathscr{B}_n]}.$$

Thus, we can express our original element $X \otimes Y \otimes Z$ as a linear combination of elements of the form $X' \otimes U'x_nV' \otimes Z'x_{n-1}$, $\pmod{k\,[\mathscr{B}_n]}$. Then using modified Hochschild etc., rewrite each such term as follows:

$$X' \otimes U'x_nV' \otimes Z'x_{n-1} \approx X'U' \otimes x_nV' \otimes Z'x_{n-1} - U' \otimes x_nV'X' \otimes Z'x_{n-1}, \pmod{k\,[\mathscr{B}_n]}.$$

By Eq. (63), we transport x_n to the end of each term, so $X' \otimes U'x_nV' \otimes Z'x_{n-1} \approx X'U' \otimes V' \otimes Z'x_{n-1}x_n - U' \otimes V'X' \otimes Z'x_{n-1}x_n$, modulo elements of $k\,[\mathscr{B}_n]$. If $|Z'| \geq 1$, then we are done. Otherwise, we have some elements of the form $X'' \otimes Y'' \otimes x_{n-1}x_n$. Use an induction argument analogous to that in step 8 to re-express this type of element as a linear combination of elements of the form $U \otimes x_{i_{n-2}} \otimes x_{n-1}x_n$ such that $i_{n-2} \neq n-2$, modulo elements of $k\,[\mathscr{B}_n]$.

Step 11. Every element of $k\,[\mathrm{Mor}_{\Delta S}\,([n],[2])]$ is equivalent to an element of $k\,[\mathscr{B}_n]$.

We shall use an iterative re-writing procedure. First of all, define sets:

$$\mathscr{B}_n^j := \{A \otimes x_{i_{n-j}} \otimes x_{n-j+1} \dots x_n \mid i_{n-j} \neq n-j\},$$

$$\mathscr{C}_n^j := \{A \otimes B \otimes Cx_{n-j+1} \dots x_n \mid |C| \geq 1\}.$$

Now clearly, $\mathscr{B}_n = \bigcup_{j=0}^{n-1} \mathscr{B}_n^j$. In what follows, 'reduced' will always mean reduced modulo elements of $k\,[\mathscr{B}_n]$. By steps 9 and 10, we can reduce an arbitrary element $X \otimes Y \otimes Z$ to linear combinations of elements in $\mathscr{B}_n^0 \cup \mathscr{B}_n^1 \cup \mathscr{B}_n^2 \cup \mathscr{C}_n^2$. Suppose now that we have reduced elements to linear combinations of elements from the set $\mathscr{B}_n^0 \cup \mathscr{B}_n^1 \cup \ldots \cup \mathscr{B}_n^j \cup \mathscr{C}_n^j$, for some $j \geq 2$. I claim any element of $\mathscr{C}_n^j$ can be re-expressed as a linear combination of elements from the set $\mathscr{B}_n^0 \cup \mathscr{B}_n^1 \cup \ldots \cup \mathscr{B}_n^{j+1} \cup \mathscr{C}_n^{j+1}$. Indeed, let $X \otimes Y \otimes Zx_{n-j+1} \dots x_n$, with $|Z| \geq 1$. Let $w := x_{n-j+1} \dots x_n$. We may now think of $X \otimes Y \otimes Zw$ as consisting of the 'indeterminates' $x_0, x_1, \ldots, x_{n-j}, w$, hence, by step 10, we may reduce this element to a linear combination of elements from the set $\{X \otimes x_{i_{n-j-1}} \otimes x_{n-j}w \mid i_{n-j-1} \neq n-j-1\} \cup \{X \otimes Y \otimes Zx_{n-j}w \mid |Z| \geq 1\}$. This implies the element may written as a linear combination of elements from the set $\mathscr{B}_n^{j+1} \cup \mathscr{C}_n^{j+1}$, modulo elements of the form $A \otimes B \otimes 1$ and $A \otimes B \otimes x_{n-j+1} \dots x_n$. Since $\{A \otimes B \otimes x_{n-j+1}x_{n-j+2} \dots x_n\} \subseteq \mathscr{C}_n^{j-1}$, the inductive hypothesis ensures that the there is set containment $\{A \otimes B \otimes x_{n-j+1} \cdots x_n\} \subseteq \mathscr{B}_n^0 \cup \ldots \cup \mathscr{B}_n^j$. This completes the inductive step.

After a finite number of iterations, then, we can re-express any element $X \otimes Y \otimes Z$ as a linear combination from the set $\mathscr{B}_n^0 \cup \ldots \mathscr{B}_n^{n-1} \cup \mathscr{C}_n^{n-1} = \mathscr{B}_n \cup \mathscr{C}_n^{n-1}$. But $\mathscr{C}_n^{n-1} = \{A \otimes B \otimes Cx_2 \dots x_n \mid |C| \geq 1\}$. Any element from this set has either $|A| = 0$ or $|B| = 0$, therefore is equivalent to an element of $k[\mathscr{B}_n]$ already. $\square$

Corollary 101. *If $\frac{1}{2} \in k$, then the four-term relation $XY \otimes Z \otimes W + W \otimes ZX \otimes Y + YZX \otimes 1 \otimes W + W \otimes YZ \otimes X \approx 0$ is sufficient to collapse $k\left[\mathrm{Mor}_{\Delta S}\left([n],[2]\right)\right]$ onto $k\left[\mathscr{B}_n\right]$.*

Proof. We only need to modify step 1 of the previous proof. We will establish that $X \otimes 1 \otimes 1 \approx 1 \otimes X \otimes 1 \approx 1 \otimes 1 \otimes X \approx 0$.

Setting three variables at a time equal to 1 in Eq. (56) we obtain,

$$2(W \otimes 1 \otimes 1) + 2(1 \otimes 1 \otimes W) \approx 0, \quad \text{when } X = Y = Z = 1. \tag{64}$$

$$Z \otimes 1 \otimes 1 + 3(1 \otimes Z \otimes 1) \approx 0, \quad \text{when } X = Y = W = 1. \tag{65}$$

$$2(Y \otimes 1 \otimes 1) + 1 \otimes Y \otimes 1 + 1 \otimes 1 \otimes Y \approx 0, \quad \text{when } X = Z = W = 1. \tag{66}$$

Equivalently, we have a system of linear equations,

$$\begin{bmatrix} 2 & 0 & 2 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = 0,$$

where $z_1 := X \otimes 1 \otimes 1$, $z_2 := 1 \otimes X \otimes 1$, and $z_3 := 1 \otimes 1 \otimes X$. Since the determinant of the coefficient matrix is -4, the matrix is invertible in the ring k as long as $1/2 \in k$. $\square$

Lemma 100 together with Lemmas 99 and 96 show the following sequence of k-modules is exact:

$$0 \leftarrow k \xleftarrow{\epsilon} k\left[\mathrm{Mor}_{\Delta S}\left([n],[0]\right)\right] \xleftarrow{\rho} k\left[\mathrm{Mor}_{\Delta S}\left([n],[2]\right)\right] \xleftarrow{(\alpha,\beta)} k\left[\mathrm{Mor}_{\Delta S}\left([n],[3]\right)\right] \oplus k\left[\mathrm{Mor}_{\Delta S}\left([n],[0]\right)\right], \tag{67}$$

where $\alpha : k\left[\mathrm{Mor}_{\Delta S}\left([n],[3]\right)\right] \to k\left[\mathrm{Mor}_{\Delta S}\left([n],[2]\right)\right]$ is given by composition with the ΔS morphism, $x_0x_1 \otimes x_2 \otimes x_3 + x_3 \otimes x_2x_0 \otimes x_1 + x_1x_2x_0 \otimes 1 \otimes x_3 + x_3 \otimes x_1x_2 \otimes x_0$, and $\beta : k\left[\mathrm{Mor}_{\Delta S}\left([n],[0]\right)\right] \to k\left[\mathrm{Mor}_{\Delta S}\left([n],[2]\right)\right]$ is induced by $1 \otimes x_0 \otimes 1$. This holds for all $n \geq 0$, so we have constructed a partial resolution of $\underline{k}$ by projective ΔS^{op}-modules:

$$0 \leftarrow k \xleftarrow{\epsilon} k\left[\mathrm{Mor}_{\Delta S}\left(-,[0]\right)\right] \xleftarrow{\rho} k\left[\mathrm{Mor}_{\Delta S}\left(-,[2]\right)\right] \xleftarrow{(\alpha,\beta)} k\left[\mathrm{Mor}_{\Delta S}\left(-,[3]\right)\right] \oplus k\left[\mathrm{Mor}_{\Delta S}\left(-,[0]\right)\right] \tag{68}$$

11. Using the Partial Resolution for Low Degree Computations

Let A be a unital associative algebra over k. Since Eq. (68) is a partial resolution of $\underline{k}$, it can be used to find $HS_i(A)$ for $i = 0, 1$.

11.1. Main Theorem.

Theorem 102. *$HS_i(A)$ for $i = 0, 1$ may be computed as the degree 0 and degree 1 homology groups of the following (partial) chain complex:*

$$0 \longleftarrow A \xleftarrow{\partial_1} A \otimes A \otimes A \xleftarrow{\partial_2} (A \otimes A \otimes A \otimes A) \oplus A, \tag{69}$$

where

$$\partial_1 : a \otimes b \otimes c \mapsto abc - cba,$$

$$\partial_2 : \begin{cases} a \otimes b \otimes c \otimes d & \mapsto & ab \otimes c \otimes d + d \otimes ca \otimes b + bca \otimes 1 \otimes d + d \otimes bc \otimes a, \\ a & \mapsto & 1 \otimes a \otimes 1. \end{cases}$$

Proof. Tensoring the complex (68) with $B_*^{sym}A$ over ΔS, we obtain a complex that computes $HS_0(A)$ and $HS_1(A)$. The statement of the theorem then follows from isomorphisms induced by the evaluation map, $k\left[\mathrm{Mor}_{\Delta S}\left(-,[p]\right)\right] \otimes_{\Delta S} B_*^{sym}A \xrightarrow{\cong} B_p^{sym}A$. $\square$

11.2. **Degree 0 Symmetric Homology.**

Theorem 103. *For a unital associative algebra A over commutative ground ring k, $HS_0(A) \cong A/([A, A])$, where $([A, A])$ is the ideal generated by the commutator submodule $[A, A]$.*

Proof. By Thm. 102, $HS_0(A) \cong A/k[\{abc - cba\}]$ as k-module. But $k\big[\{abc - cba\}\big]$ is an ideal of A. Now clearly $[A, A] \subseteq k[\{abc - bca\}]$. On the other hand, $k[\{abc - cba\}] \subseteq ([A, A])$ since $abc - cba = a(bc - cb) + a(cb) - (cb)a$. □

Corollary 104. *If A is commutative, then $HS_0(A) \cong A$.*

Remark 105. Theorem 103 implies that symmetric homology does not preserve Morita equivalence, since for $n > 1$, $HS_0(M_n(A)) = M_n(A)/([M_n(A), M_n(A)]) = 0$, while in general $HS_0(A) = A/([A, A]) \neq 0$.

11.3. **Degree 1 Symmetric Homology.** Using `GAP`, we have made the following explicit computations of degree 1 integral symmetric homology. See section 13 for a discussion of how computer algebra systems were used in symmetric homology computations.

A	$HS_1(A \mid \mathbb{Z})$
$\mathbb{Z}[t]/(t^2)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
$\mathbb{Z}[t]/(t^3)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
$\mathbb{Z}[t]/(t^4)$	$(\mathbb{Z}/2\mathbb{Z})^4$
$\mathbb{Z}[t]/(t^5)$	$(\mathbb{Z}/2\mathbb{Z})^4$
$\mathbb{Z}[t]/(t^6)$	$(\mathbb{Z}/2\mathbb{Z})^6$
$\mathbb{Z}[C_2]$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
$\mathbb{Z}[C_3]$	0
$\mathbb{Z}[C_4]$	$(\mathbb{Z}/2\mathbb{Z})^4$
$\mathbb{Z}[C_5]$	0
$\mathbb{Z}[C_6]$	$(\mathbb{Z}/2\mathbb{Z})^6$

Based on these calculations, we conjecture:

Conjecture 106.

$$HS_1\big(k[t]/(t^n)\big) = \begin{cases} (k/2k)^n, & \text{if } n \geq 0 \text{ is even.} \\ (k/2k)^{n-1} & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

Remark 107. The computations of $HS_1\big(\mathbb{Z}[C_n]\big)$ are consistent with those of Brown and Loday [2]. See 11.5 for a more detailed treatment of HS_1 for group rings.

Additionally, HS_1 has been computed for the following examples. These computations were done using `GAP` in some cases and in others, `Fermat` [11] computations on sparse matrices were used in conjunction with the `GAP` scripts. (*e.g.* when the algebra has dimension greater than 6 over $\mathbb{Z}$).

A	$HS_1(A \mid \mathbb{Z})$
$\mathbb{Z}[t,u]/(t^2, u^2)$	$\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{11}$
$\mathbb{Z}[t,u]/(t^3, u^2)$	$\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^{11} \oplus \mathbb{Z}/6\mathbb{Z}$
$\mathbb{Z}[t,u]/(t^3, u^2, t^2u)$	$\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$
$\mathbb{Z}[t,u]/(t^3, u^3)$	$\mathbb{Z}^4 \oplus (\mathbb{Z}/2\mathbb{Z})^{7} \oplus (\mathbb{Z}/6\mathbb{Z})^5$
$\mathbb{Z}[t,u]/(t^2, u^4)$	$\mathbb{Z}^3 \oplus (\mathbb{Z}/2\mathbb{Z})^{20} \oplus \mathbb{Z}/4\mathbb{Z}$
$\mathbb{Z}[t,u,v]/(t^2, u^2, v^2)$	$\mathbb{Z}^6 \oplus (\mathbb{Z}/2\mathbb{Z})^{42}$
$\mathbb{Z}[t,u]/(t^4, u^3)$	$\mathbb{Z}^6 \oplus (\mathbb{Z}/2\mathbb{Z})^{19} \oplus \mathbb{Z}/6\mathbb{Z} \oplus (\mathbb{Z}/12\mathbb{Z})^2$
$\mathbb{Z}[t,u,v]/(t^2, u^2, v^3)$	$\mathbb{Z}^{11} \oplus (\mathbb{Z}/2\mathbb{Z})^{45} \oplus (\mathbb{Z}/6\mathbb{Z})^4$
$\mathbb{Z}[i,j,k], i^2 = j^2 = k^2 = ijk = -1$	$(\mathbb{Z}/2\mathbb{Z})^8$
$\mathbb{Z}[C_2 \times C_2]$	$(\mathbb{Z}/2\mathbb{Z})^{12}$
$\mathbb{Z}[C_3 \times C_2]$	$(\mathbb{Z}/2\mathbb{Z})^{6}$
$\mathbb{Z}[C_3 \times C_3]$	$(\mathbb{Z}/3\mathbb{Z})^{9}$
$\mathbb{Z}[S_3]$	$(\mathbb{Z}/2\mathbb{Z})^{2}$

11.4. **Splittings of the Partial Complex.** Under certain circumstances, the partial complex in Thm.102 splits as a direct sum of smaller complexes. This observation becomes increasingly important as the dimension of the algebra increases. Indeed, some of the computations of the previous section were done using splittings.

Definition 108. For a commutative k-algebra A and $u \in A$, define the k-modules:

$$\left(A^{\otimes n}\right)_u := \{a_1 \otimes a_2 \otimes \ldots \otimes a_n \in A^{\otimes n} \mid a_1 a_2 \cdot \ldots \cdot a_n = u\}$$

Proposition 109. *If $A = k[M]$ for a commutative monoid M, then complex (69) splits as a direct sum of complexes*

$$0 \longleftarrow (A)_u \xleftarrow{\partial_1} (A \otimes A \otimes A)_u \xleftarrow{\partial_2} (A \otimes A \otimes A \otimes A)_u \oplus (A)_u, \tag{70}$$

where u ranges over the elements of M. Thus, for $i = 0, 1$, we have $HS_i(A) \cong \bigoplus_{u \in M} HS_i(A)_u$.

Proof. Since M is a commutative monoid, there are direct sum decompositions as k-module: $A^{\otimes n} = \bigoplus_{u \in M} \left(A^{\otimes n}\right)_u$. The boundary maps ∂_1 and ∂_2 preserve the products of tensor factors, so the inclusions $\left(A^{\otimes n}\right)_u \hookrightarrow A^{\otimes n}$ induce maps of complexes, hence the complex itself splits as a direct sum. □

Definition 110. For each u, the homology groups of complex (70) will be called the *u-layered symmetric homology* of A, denoted $HS_i(A)_u$.

We may use layers to investigate the symmetric homology of $k[t]$. This algebra is monoidal, generated by the monoid $\{1, t, t^2, t^3, \ldots\}$. Now, the t^m-layer symmetric homology of $k[t]$ will be the same as the t^m-layer symmetric homology of $k\left[M_{m+1}^{m+2}\right]$, where M_q^p denotes the cyclic monoid generated by an indeterminate s with the property that $s^p = s^q$. Using this observation and subsequent computation, we conjecture:

Conjecture 111.

$$HS_1\left(k[t]\right)_{t^m} = \begin{cases} 0 & m = 0, 1 \\ k/2k, & m \geq 2 \end{cases}$$

This conjecture has been verified up to $m = 18$, in the case $k = \mathbb{Z}$.

11.5. **2-torsion in HS_1.** The occurrence of 2-torsion in $HS_1(A)$ for the examples considered in 11.3 and 11.4 comes as no surprise, based on Thm. 44. First consider the following chain of isomorphisms: $\pi_2^s(B\Gamma) = \pi_2\left(\Omega^\infty S^\infty(B\Gamma)\right) \cong \pi_1\left(\Omega\Omega^\infty S^\infty(B\Gamma)\right) \cong \pi_1\left(\Omega_0\Omega^\infty S^\infty(B\Gamma)\right) \xrightarrow{h} H_1\left(\Omega_0\Omega^\infty S^\infty(B\Gamma)\right)$. Here, $\Omega_0\Omega^\infty S^\infty(B\Gamma)$ denotes the component of the constant loop, and h is the Hurewicz homomorphism, which is an isomorphism since $\Omega_0\Omega^\infty S^\infty(B\Gamma)$ is path-connected and π_1 is abelian (since it is actually π_2 of a space).

On the other hand, by Thm. 44, $HS_1(k[\Gamma]) \cong H_1\left(\Omega\Omega^\infty S^\infty(B\Gamma); k\right) \cong H_1\left(\Omega\Omega^\infty S^\infty(B\Gamma)\right) \otimes k$. Note, all tensor products will be over $\mathbb{Z}$ in this section. Now $\Omega\Omega^\infty S^\infty(B\Gamma)$ consists of disjoint homeomorphic copies of $\Omega_0\Omega^\infty S^\infty(B\Gamma)$, one for each element of $\Gamma/[\Gamma, \Gamma]$, (where $[\Gamma, \Gamma]$ is the commutator subgroup of Γ), so we may write $H_1\left(\Omega\Omega^\infty S^\infty(B\Gamma)\right) \otimes k \cong H_1\left(\Omega_0\Omega^\infty S^\infty(B\Gamma)\right) \otimes k\left[\Gamma/[\Gamma, \Gamma]\right]$ and obtain the following result:

Proposition 112. *If Γ is a group, then $HS_1(k[\Gamma]) \cong \pi_2^s(B\Gamma) \otimes k\left[\Gamma/[\Gamma, \Gamma]\right]$.*

As an immediate corollary, if Γ is abelian, then $HS_1(k[\Gamma]) \cong \pi_2^s(B\Gamma) \otimes k[\Gamma]$. Moreover, by results of Brown and Loday [2], if Γ is abelian, then $\pi_2^s(B\Gamma)$ is the reduced tensor square, $\Gamma \mathbin{\widetilde{\wedge}} \Gamma = (\Gamma \otimes \Gamma) / \approx$, where $g \otimes h \approx -h \otimes g$ for all $g, h \in \Gamma$. (This construction is notated with multiplicative group action in [2], since they deal with the more general case of non-abelian groups.)

Proposition 113.

$$HS_1(k[C_n]) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^n & n \text{ even.} \\ 0 & n \text{ odd.} \end{cases}$$

Proof. $\pi_2^s(BC_n) = \mathbb{Z}/2\mathbb{Z}$ if n is even, and 0 if n is odd. The result then follows from Prop. 112, as $k\left[C_n/[C_n, C_n]\right] \cong k[C_n] \cong k^n$, as k-module. □

12. Relations to Cyclic Homology

The relation between the symmetric bar construction and the cyclic bar construction arising from the inclusions $\Delta C \hookrightarrow \Delta S$ gives rise to a natural map $HC_*(A) \to HS_*(A)$ Indeed, by remark 13, we may define cyclic homology thus: $HC_*(A) = \mathrm{Tor}_*^{\Delta C}(\underline{k}, B_*^{sym}A)$, where we understand $B_*^{sym}A$ as the restriction of the functor to ΔC.

Using the partial complex of Thm. 102, and an analogous one for computing cyclic homology (c.f. [12], p. 59), the map $HC_*(A) \to HS_*(A)$ for degrees 0 and 1 is induced by the following partial chain map:

$$\begin{tikzcd} 0 & A \arrow[l] \arrow[d, "\gamma_0=\mathrm{id}"] & A\otimes A \arrow[l, "\partial_1^C"'] \arrow[d, "\gamma_1"] & A^{\otimes 3}\oplus A \arrow[l, "\partial_2^C"'] \arrow[d, "\gamma_2"] \\ 0 & A \arrow[l] & A^{\otimes 3} \arrow[l, "\partial_1^S"'] & A^{\otimes 4}\oplus A \arrow[l, "\partial_2^S"'] \end{tikzcd}$$

In this diagram, the boundary maps in the upper row are defined as follows:

$$\partial_1^C : a\otimes b \mapsto ab - ba$$

$$\partial_2^C : \begin{cases} a\otimes b\otimes c & \mapsto ab\otimes c - a\otimes bc + ca\otimes b \\ a & \mapsto 1\otimes a - a\otimes 1 \end{cases}$$

The boundary maps in the lower row are defined as in Thm. 102.

$$\partial_1^S : a\otimes b\otimes c \mapsto abc - cba$$

$$\partial_2^S : \begin{cases} a\otimes b\otimes c\otimes d & \mapsto ab\otimes c\otimes d - d\otimes ca\otimes b + bca\otimes 1\otimes d + d\otimes bc\otimes a \\ a & \mapsto 1\otimes a\otimes 1 \end{cases}$$

The partial chain map is given in degree 1 by $\gamma_1(a\otimes b) := a\otimes b\otimes 1$. In degree 2, γ_2 is defined on the summand $A^{\otimes 3}$ via

$$a\otimes b\otimes c \mapsto (a\otimes b\otimes c\otimes 1 - 1\otimes a\otimes bc\otimes 1 + 1\otimes ca\otimes b\otimes 1 + 1\otimes 1\otimes abc\otimes 1 - b\otimes ca\otimes 1\otimes 1) - 2abc - cab,$$

and on the summand A via

$$a \mapsto (-1\otimes 1\otimes a\otimes 1) + (4a).$$

12.1. **Examples.** To provide some examples, consider the maps $\gamma_1 : HC_1(\mathbb{Z}[t]/(t^n)) \to HS_1(\mathbb{Z}[t]/(t^n))$.

It can be shown (*e.g.* by direct computation) that $HC_1(\mathbb{Z}[t]/(t^2)) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the 1-chain $t\otimes t$. $\gamma_1(t\otimes t) = t\otimes t\otimes 1 \in HS_1(\mathbb{Z}[t]/(t^2)$ is a non-trivial element of $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ (which may be verified by direct computation as well).

The map $HC_1(\mathbb{Z}[t]/(t^3)) \to HS_1(\mathbb{Z}[t]/(t^2))$ may be similarly analyzed. Here, the chain $t\otimes t + t\otimes t^2$ is a generator of $HC_1(\mathbb{Z}[t]/(t^3)) \cong \mathbb{Z}/6\mathbb{Z}$, which gets sent by γ_1 to $t\otimes t\otimes 1 + t\otimes t^2\otimes 1$, a non-trivial element of $HS_1(\mathbb{Z}[t]/(t^3)) \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.

The case $n = 4$ is bit more interesting. Here, $HC_1(\mathbb{Z}[t]/(t^4)) \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/12\mathbb{Z}$, generated by $t\otimes t$ and $t\otimes t^2 + t\otimes t^3$, respectively. The image of the map γ_1 in $HS_1(\mathbb{Z}[t]/(t^4))$ is $(\mathbb{Z}/2\mathbb{Z})^2 \subseteq (\mathbb{Z}/2\mathbb{Z})^4$.

13. Using Computer Algebra Systems for Computing Symmetric Homology

The computer algebra systems `GAP`, `Octave` and `Fermat` were used to verify proposed theorems and also to obtain some concrete computations of symmetric homology for some small algebras. A tar-file of the scripts that were created and used for this work is available at `http://arxiv.org/e-print/0807.4521v1/`. This tar-file contains the following files:

- `Basic.g` - Some elementary functions, necessary for some functions in `DeltaS.g`
- `HomAlg.g` - Homological Algebra functions, such as computation of homology groups for chain complexes.
- `Fermat.g` - Functions necessary to invoke `Fermat` for fast sparse matrix computations.
- `fermattogap`, `gaptofermat` - Auxiliary text files for use when invoking `Fermat` from `GAP`.
- `DeltaS.g` - This is the main repository of scripts used to compute various quantities associated with the category ΔS, including $HS_1(A)$ for finite-dimensional algebras A.

In order to use the functions of `DeltaS.g`, simply copy the above files into the working directory (such as `~/gap/`), invoke `GAP`, then read in `DeltaS.g` at the prompt. The dependent modules will automatically be loaded (hence they must be present in the same directory as `DeltaS.g`). Note, most of the computations involving homology require substantial memory to run. I recommend calling `GAP` with the command line option "`-o` *mem*", where *mem* is the amount of memory to be allocated to this instance of `GAP`. All computations done in this dissertation can be accomplished by allocating 20 gigabytes of memory. The following provides a few examples of using the functions of `DeltaS.g`

```
[ault@math gap]$ gap -o 20g

gap> Read("DeltaS.g");
gap>
gap> ## Number of morphisms [6] --> [4]
gap> SizeDeltaS( 6, 4 );
1663200
gap>
gap> ## Generate the set of morphisms of Delta S, [2] --> [2]
gap> EnumerateDeltaS( 2, 2 );
[ [ [ 0, 1, 2 ], [  ], [  ] ], [ [ 0, 2, 1 ], [  ], [  ] ],
  [ [ 1, 0, 2 ], [  ], [  ] ], [ [ 1, 2, 0 ], [  ], [  ] ],
  [ [ 2, 0, 1 ], [  ], [  ] ], [ [ 2, 1, 0 ], [  ], [  ] ],
  [ [ 0, 1 ], [ 2 ], [  ] ], [ [ 0, 2 ], [ 1 ], [  ] ],
  [ [ 1, 0 ], [ 2 ], [  ] ], [ [ 1, 2 ], [ 0 ], [  ] ],
  [ [ 2, 0 ], [ 1 ], [  ] ], [ [ 2, 1 ], [ 0 ], [  ] ],
  [ [ 0, 1 ], [  ], [ 2 ] ], [ [ 0, 2 ], [  ], [ 1 ] ],
  [ [ 1, 0 ], [  ], [ 2 ] ], [ [ 1, 2 ], [  ], [ 0 ] ],
  [ [ 2, 0 ], [  ], [ 1 ] ], [ [ 2, 1 ], [  ], [ 0 ] ],
  [ [ 0 ], [ 1, 2 ], [  ] ], [ [ 0 ], [ 2, 1 ], [  ] ],
  [ [ 1 ], [ 0, 2 ], [  ] ], [ [ 1 ], [ 2, 0 ], [  ] ],
  [ [ 2 ], [ 0, 1 ], [  ] ], [ [ 2 ], [ 1, 0 ], [  ] ],
  [ [ 0 ], [ 1 ], [ 2 ] ], [ [ 0 ], [ 2 ], [ 1 ] ], [ [ 1 ], [ 0 ], [ 2 ] ],
  [ [ 1 ], [ 2 ], [ 0 ] ], [ [ 2 ], [ 0 ], [ 1 ] ], [ [ 2 ], [ 1 ], [ 0 ] ],
  [ [ 0 ], [  ], [ 1, 2 ] ], [ [ 0 ], [  ], [ 2, 1 ] ],
  [ [ 1 ], [  ], [ 0, 2 ] ], [ [ 1 ], [  ], [ 2, 0 ] ],
  [ [ 2 ], [  ], [ 0, 1 ] ], [ [ 2 ], [  ], [ 1, 0 ] ],
  [ [  ], [ 0, 1, 2 ], [  ] ], [ [  ], [ 0, 2, 1 ], [  ] ],
  [ [  ], [ 1, 0, 2 ], [  ] ], [ [  ], [ 1, 2, 0 ], [  ] ],
  [ [  ], [ 2, 0, 1 ], [  ] ], [ [  ], [ 2, 1, 0 ], [  ] ],
  [ [  ], [ 0, 1 ], [ 2 ] ], [ [  ], [ 0, 2 ], [ 1 ] ],
  [ [  ], [ 1, 0 ], [ 2 ] ], [ [  ], [ 1, 2 ], [ 0 ] ],
  [ [  ], [ 2, 0 ], [ 1 ] ], [ [  ], [ 2, 1 ], [ 0 ] ],
  [ [  ], [ 0 ], [ 1, 2 ] ], [ [  ], [ 0 ], [ 2, 1 ] ],
  [ [  ], [ 1 ], [ 0, 2 ] ], [ [  ], [ 1 ], [ 2, 0 ] ],
  [ [  ], [ 2 ], [ 0, 1 ] ], [ [  ], [ 2 ], [ 1, 0 ] ],
  [ [  ], [  ], [ 0, 1, 2 ] ], [ [  ], [  ], [ 0, 2, 1 ] ],
  [ [  ], [  ], [ 1, 0, 2 ] ], [ [  ], [  ], [ 1, 2, 0 ] ],
  [ [  ], [  ], [ 2, 0, 1 ] ], [ [  ], [  ], [ 2, 1, 0 ] ] ]
gap>
gap> ## Generate only the epimorphisms [2] -->> [2]
gap> EnumerateDeltaS( 2, 2 : epi );
[ [ [ 0 ], [ 1 ], [ 2 ] ], [ [ 0 ], [ 2 ], [ 1 ] ],
  [ [ 1 ], [ 0 ], [ 2 ] ], [ [ 1 ], [ 2 ], [ 0 ] ],
  [ [ 2 ], [ 0 ], [ 1 ] ], [ [ 2 ], [ 1 ], [ 0 ] ] ]
gap>
gap> ## Compose two morphisms of Delta S.
```

```
gap> a := Random(EnumerateDeltaS(4,3));
[ [ 0 ], [ 2, 4, 1 ], [  ], [ 3 ] ]
gap> b := Random(EnumerateDeltaS(3,2));
[ [  ], [ 3, 0, 2 ], [ 1 ] ]
gap> MultDeltaS(b, a);
[ [  ], [ 3, 0 ], [ 2, 4, 1 ] ]
gap> MultDeltaS(a, b);
Maps incomposeable
[  ]
gap>
gap> ## Examples of using morphisms of Delta S to act on simple tensors
gap> A := TruncPolyAlg([3,2]);
<algebra of dimension 6 over Rationals>
gap> ## TruncPolyAlg is defined in Basic.g
gap> ##  TruncPolyAlg([i_1, i_2, ..., i_n]) is generated by
gap> ##  x_1, x_2, ..., x_n, under the relation (x_j)^(i_j) = 0.
gap> g := GeneratorsOfLeftModule(A);
[ X^[ 0, 0 ], X^[ 0, 1 ], X^[ 1, 0 ], X^[ 1, 1 ], X^[ 2, 0 ], X^[ 2, 1 ] ]
gap> x := g[2]; y := g[3];
X^[ 0, 1 ]
X^[ 1, 0 ]
gap> v := [ x*y, 1, y^2 ];
gap> ## v represents the simple tensor  xy \otimes 1 \otimes y^2.
[ X^[ 1, 1 ], 1, X^[ 2, 0 ] ]
gap> ActByDeltaS( v, [[2], [], [0], [1]] );
[ X^[ 2, 0 ], 1, X^[ 1, 1 ], 1 ]
gap> ActByDeltaS( v, [[2], [0,1]] );
[ X^[ 2, 0 ], X^[ 1, 1 ] ]
gap> ActByDeltaS( v, [[2,0], [1]] );
[ 0*X^[ 0, 0 ], 1 ]
gap>
gap> ## Symmetric monoidal product on DeltaS_+
gap> a := Random(EnumerateDeltaS(4,2));
[ [  ], [ 2, 1, 0 ], [ 3, 4 ] ]
gap> b := Random(EnumerateDeltaS(3,3));
[ [  ], [  ], [  ], [ 1, 3, 2, 0 ] ]
gap> MonoidProductDeltaS(a, b);
[ [  ], [ 2, 1, 0 ], [ 3, 4 ], [  ], [  ], [  ], [ 6, 8, 7, 5 ] ]
gap> MonoidProductDeltaS(b, a);
[ [  ], [  ], [  ], [ 1, 3, 2, 0 ], [  ], [ 6, 5, 4 ], [ 7, 8 ] ]
gap> MonoidProductDeltaS(a, []);
[ [  ], [ 2, 1, 0 ], [ 3, 4 ] ]
gap>
gap> ## Symmetric Homology of the algebra A, in degrees 0 and 1.
gap> SymHomUnitalAlg(A);
[ [ 0, 0, 0, 0, 0, 0 ], [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 6, 0, 0 ] ]
gap> ## '0' represents a factor of Z, while a non-zero p represents
gap> ## a factor of Z/pZ.
gap>
gap> ## Using layers to compute symmetric homology
gap> C2 := CyclicGroup(2);
<pc group of size 2 with 1 generators>
gap> A := GroupRing(Rationals, DirectProduct(C2, C2));
<algebra-with-one over Rationals, with 2 generators>
```

```
gap> ## First, a direct computation without layers:
gap> SymHomUnitalAlg(A);
[ [ 0, 0, 0, 0 ], [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ] ]
gap> ## Next, compute HS_0(A)_u and HS_1(A)_u for each generator u.
gap> g := GeneratorsOfLeftModule(A);
[ (1)*<identity> of ..., (1)*f2, (1)*f1, (1)*f1*f2 ]
gap> SymHomUnitalAlgLayered(A, g[1]);
[ [ 0 ], [ 2, 2, 2 ] ]
gap> SymHomUnitalAlgLayered(A, g[2]);
[ [ 0 ], [ 2, 2, 2 ] ]
gap> SymHomUnitalAlgLayered(A, g[3]);
[ [ 0 ], [ 2, 2, 2 ] ]
gap> SymHomUnitalAlgLayered(A, g[4]);
[ [ 0 ], [ 2, 2, 2 ] ]
gap> ## Computing HS_1( Z[t] ) by layers:
gap> SymHomFreeMonoid(0,10);
HS_1(k[t])_{t^0} :  [  ]
HS_1(k[t])_{t^1} :  [  ]
HS_1(k[t])_{t^2} :  [ 2 ]
HS_1(k[t])_{t^3} :  [ 2 ]
HS_1(k[t])_{t^4} :  [ 2 ]
HS_1(k[t])_{t^5} :  [ 2 ]
HS_1(k[t])_{t^6} :  [ 2 ]
HS_1(k[t])_{t^7} :  [ 2 ]
HS_1(k[t])_{t^8} :  [ 2 ]
HS_1(k[t])_{t^9} :  [ 2 ]
HS_1(k[t])_{t^10} :  [ 2 ]
gap> ## Poincare polynomial of Sym_*^{(p)} for small p.
gap> ##  There is a check for torsion, using a call to Fermat
gap> ##  to find Smith Normal Form of the differential matrices.
gap> PoincarePolynomialSymComplex(2);
C_0  Dimension: 1
C_1  Dimension: 6
C_2  Dimension: 6
D_1
SNF(D_1)
D_2
SNF(D_2)
2*t^2+t
gap> PoincarePolynomialSymComplex(5);
C_0  Dimension: 1
C_1  Dimension: 30
C_2  Dimension: 300
C_3  Dimension: 1200
C_4  Dimension: 1800
C_5  Dimension: 720
D_1
SNF(D_1)
D_2
SNF(D_2)
D_3
SNF(D_3)
D_4
SNF(D_4)
```

```
D_5
SNF(D_5)
120*t^5+272*t^4+t^3
```

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Department of Mathematics, Fordham University, Bronx, New York, 10461, USA.
E-mail address: `ault@fordham.edu`